Paths and Principle of Maupertuis

→ Abbreviated Action / Principle of Maupertuis

Now, $\delta S = 0$ (Principle Least Action) ⇒ { "Path" / Position ↕ trajectory }

Position : $\underline{q}(t)$

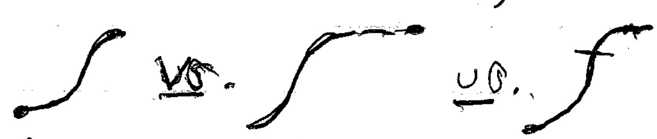
Path : $\underline{r}(\ell)$ → curve followed by particle (but not when particle at particular point) (i.e. geodesic)

$$\partial_t L = 0 \Rightarrow H(p,q) = E$$

Now $$\delta \int_{q_1, t_1}^{q_2, t_2} L = 0$$ fixed endpoints ⇒ $\delta S = 0$

but if allow $\underline{\underline{t_2}}$ to vary: (virtual paths $q_1 \to q_2$ but t variable)

$$\delta \int_{q_1, t_1}^{q_2, t} L = -H \delta t$$



i.e. particle passes thru q_2, but not necessarily at $\underline{\underline{t_2}}$.
↓ "path"

(i.e. $dS = \int p\, dq - \int H\, dt$)

or, for energy conserving virtual paths:

$$\delta S + E\,\delta t = 0$$

Now also: $S = \int \sum_i p_i \, dq_i - E(t - t_0)$

$$\Downarrow$$

$$S_0 = \int \sum_i p_i \, dq_i \equiv \text{abbreviated action}$$

$\Rightarrow$ for paths:

$$\boxed{\delta S_0 = \delta \int \sum_i p_i \, dq_i = 0}$$

Principle of Maupertuis

$\rightarrow$ abbreviated action has minimum with respect to all paths which conserve energy and pass thru final point at any t.

$\rightarrow$ to use, need express momenta in terms q, dq, via

$$p_i = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q})$$

$$E(q, \dot{q}) = E$$

i.e.

$$L = \frac{1}{2} \sum_{i,k} a_{ik}(q)\, \dot{q}_i \dot{q}_k - U(q)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_k a_{ik}(q)\, \dot{q}_k$$

so

$$E = \frac{1}{2} \sum_{i,k} a_{ik}(q)\, \dot{q}_i \dot{q}_k + U(q)$$

$$\Rightarrow \quad E - U = \frac{1}{2} \sum_{i,k} a_{ik}(q) \frac{dq_i\, dq_k}{(dt)^2}$$

$$\therefore \quad dt = \left(\sum a_{ik}\, dq_i\, dq_k \Big/ 2(E-U) \right)^{1/2}$$

Thus, can write:

$$dS_0 = \sum_i p_i\, dq_i$$

$$= \sum_k a_{ik}(q)\, \dot{q}_k\, dq_i = \sum_k a_{ik}(q) \frac{dq_k}{dt}\, dq_i$$

plugging in dt $\Rightarrow$

$$\Rightarrow \boxed{S_0 = \int \left[2(E-U) \sum_{i,k} a_{ik}\, dq_i\, dq_k \right]^{1/2}}$$

$\rightarrow$ Variational for <u>Path</u>

For single particle: $T = \frac{1}{2} m \left(d\ell/dt\right)^2$

$\underbrace{d\ell}_{\text{path element}}$

$$\Rightarrow \boxed{\delta S_0 = \delta \int_{q_1}^{q_2} \left[2m(E-U)\right] d\ell = 0}$$

- Jacobi's Integral
- $U = 0 \Rightarrow \delta S_0 = \delta \int d\ell = 0$

Path of Least Action is Geodesic!

N.B. Can get orbit from dt eqn.

Example: Differential Eqn. for Path?

$$\delta \int (E-U)^{1/2} d\ell$$

$$= -\left[\int \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2(E-U)^{1/2}} d\ell - (E-U)^{1/2} d\,\delta\ell \right]$$

but $dl^2 = dr^2$

$$dl\, d\delta l = d\underline{r} \cdot d\,\delta\underline{r}$$

$$d\,\delta l = \frac{d\underline{r}}{dl} \cdot d\delta\underline{r}$$

$\Rightarrow$

$$\delta\int\left(\sqrt{E-U}\right) dl =$$

$$-\int\left\{\frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta\underline{r}}{2\sqrt{E-U}}\, dl - \sqrt{(E-U)}\,\frac{d\underline{r}}{dl} \cdot d\delta\underline{r}\right\}$$

IBP

$$0 = -\int\left\{\frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta\underline{r}}{2\sqrt{E-U}}\, dl + \frac{d}{dl}\left[\sqrt{E-U}\,\frac{d\underline{r}}{dl}\right] \cdot \delta\underline{r}\, dl\right\}$$

$$\Rightarrow \boxed{2\,(E-U)^{1/2}\,\frac{d}{dl}\left[\sqrt{E-U}\,\frac{d\underline{r}}{dl}\right] = -\frac{\partial U}{\partial \underline{r}}}$$

$\rightarrow$ equation for path !

N.B.: For eikonal theory:

$$\delta S_0 \Rightarrow \delta \Phi_0 = \delta \int \underline{k} \cdot d\underline{x}$$

eqn. for ray path. Need eliminate $\underline{k}$ in terms ω, $n(\underline{x})$, etc, to actually obtain equation for ray.

Summary – Variational Principles of Mechanics

i.) ('Standard') Principle Least Action

$$\delta \int_{q_1, t_1}^{q_2, t_2} dt\, L = 0$$

(fixed e.p.)

$\Rightarrow$ Lagrange Eqns.
Hamilton Eqns.
Liouville's Thm.

$\leftrightarrow$ trajectory, phase space flow

ii.) $$\delta \int_{q_1, t_1}^{q, t} L\, dt = \delta S \qquad S(q, t) \quad \left\{ \begin{array}{l} \text{upper E.P.} \\ \text{Variable} \end{array} \right.$$

$$\Rightarrow \frac{\partial S}{\partial t} + H(\nabla_q S, q, t) = 0$$

Hamilton–Jacobi Theory

$\leftrightarrow$ integrability, especially in different geometries.

iii.) $$\delta S_0 = \delta \int p\, dq = 0 \qquad \left\{ \begin{array}{l} \text{no time} \\ \text{specified} \end{array} \right.$$

$\Rightarrow$ path equation – curve of trajectory with no time specification

$\leftrightarrow$ ray paths, etc., in optics.