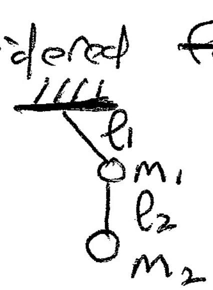
General Continuum Dynamics

→ An Introduction

→ Small Oscillations II — { Chains, Strings and the Transition Discrete → Continuous

— previously, considered few-degree-of-freedom systems, i.e.



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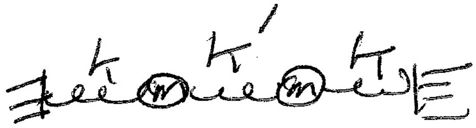
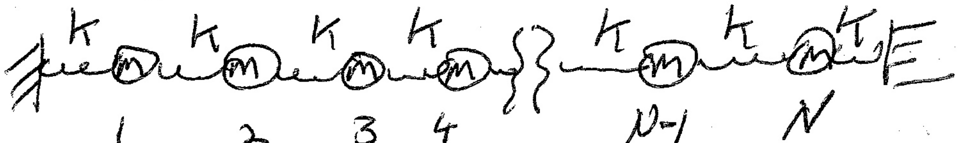


now, consider systems with $N \gg 1$ degrees of freedom, i.e.

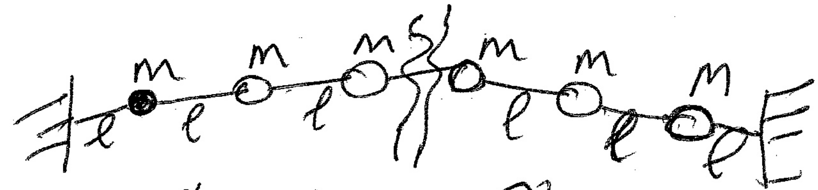
(separated by l at equilibrium)

i) — linear chain (1D oscillators)



(identical components) → monatomic

ii) — massless string (loaded)



uniform tension T

uniform { mass m, separation l

For i)

$$\frac{1}{2}k(x_{i+1}-x_i)^2$$

$$L=\sum_{i=1}^{N}\left(\frac{1}{2}m\dot{x}_i^2-\left(\frac{1}{2}k(x_i-x_{i-1})^2+\frac{1}{2}k(x_{i+1}-x_i)^2\right)\right)$$

$$\begin{cases} x_0 \equiv 0 \\ x_{N+1} \equiv 0 \end{cases}$$

or simply

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m \dot{x}_i^2 - \frac{k}{2} (x_{i+1} - x_i)^2 \right)$$

{ Compressional Modes

For ii.),

$$L = \sum_{i=1} \left(\frac{1}{2} m \dot{y}_i^2 - \frac{T}{2\ell} (y_{i+1} - y_i)^2 \right)$$

{ transverse modes

$\cong$ identical systems.

$\therefore$ hereafter, focus on i.)

motivations for i.) — monatomic chain is simplest example of elastic wave in solid

— step toward continuous system
i.e. now discrete $\rightarrow$ masses separated by ℓ

Proceeding:

$$m \ddot{x}_i - k \left[(x_{i+1} - x_i) + (x_{i-1} - x_i) \right] = 0$$

$$\ddot{x}_i + \frac{k}{m} \left[2x_i - (x_{i+1} + x_{i-1}) \right] = 0$$

$$x_i = \hat{x}_i e^{-i\omega t}$$

$$\left(\frac{2k}{m}-\omega^2\right)\hat{x}_i - \frac{k}{m}\left(\hat{x}_{i-1}+\hat{x}_{i+1}\right) = 0$$

For eigenvalues, det $\underline{\underline{A}} = 0$

$$\underline{\underline{A}} = \begin{vmatrix} \frac{2k}{m}-\omega^2 & -k/m & & & \\ -k/m & \frac{2k}{m}-\omega^2 & -k/m & & 0 \\ & -k/m & \frac{2k}{m}-\omega^2 & -k/m & \\ 0 & & -k/m & \frac{2k}{m}-\omega^2 & -k/m \end{vmatrix}$$

i.e. A tri-diagonal.

Now, taking masses separated by l, take

$$\hat{x}_n \sim e^{i(nl)\underbrace{\alpha}_{\text{wave-vector}}} \qquad \left\{\begin{array}{l} n \equiv \text{bead \#} \\ \alpha \equiv \text{wave \#} \\ l \equiv \text{spacing} \end{array}\right.$$

$\Rightarrow$

$$\left(\frac{2k}{m}-\omega^2\right)e^{i[i\, l\alpha]} - \frac{k}{m}\left(e^{i[(i+1)\, l\alpha]} + e^{i[(i-1)\, l\alpha]}\right) = 0$$

$$\therefore \quad \left(\frac{2k}{m}-\omega^2\right) - \frac{2k}{m}\cos\left[(\alpha l)\right] = 0$$

sol/ $\omega^2 = \frac{2k}{m}(2)\left[\frac{1-\cos(\alpha\ell)}{2}\right]$

$= \frac{4k}{m}\sin^2\left(\frac{\alpha\ell}{2}\right)$

$\Rightarrow$ $\boxed{\omega^2 = \frac{4k}{m}\sin^2(\alpha\ell/2)}$

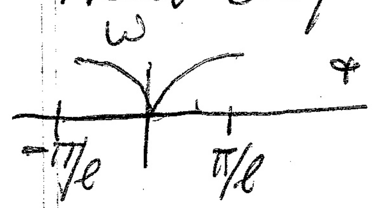
$\omega = 2\sqrt{k/m}\,|\sin\alpha\ell/2|$

Note:

① − $\omega = \omega_{max}\,|\sin\ell\alpha/2|$; $\omega_{max}^2 = 4k/m$

$\begin{cases} \omega(\alpha) = \omega(-\alpha) \\ \alpha' = \alpha + 2\pi/\ell \end{cases}$ leaves ω invariant

i.e. need only define α on $[-\pi/\ell, \pi/\ell]$ i.e. First Brillouin Zone, only needed



& repeats

② − for $\ell\alpha/2 \ll 1$

i.e. wavelength $\alpha^{-1} \gg$ bead spacing ℓ

$\rightarrow$ continuum limit

then $\omega = \sqrt{k/m}\, \ell\, \alpha$

$$= \alpha \left[\ell \sqrt{k/m} \right]$$

akin to acoustic wave

$$\omega = k\, c_s$$

$k \leftrightarrow \alpha$

$c_s \leftrightarrow \ell \sqrt{k/m}$

$\dfrac{\underline{\gamma p}}{\rho} \leftrightarrow \dfrac{\underline{\ell^2 k}}{M}$ → stored elastic energy (springiness)

→ inertia

③ – observe $\underline{\text{maximum}}$ frequency propagated is:

$$\omega^2 = \omega_{max}^2 = 4k/m$$

i.e. $\omega^2 > \omega_{max}^2$ not propagated

$\omega^2 < \omega_{max}^2$ propagated

Chain acts as low-pass filter

Higher frequencies evanescent!

④ – for propagation structure;

$$\omega = 2\sqrt{k/m}\, \left[\sin \alpha \ell/2\right]$$

$$V_{gr} = d\omega/d\alpha = \ell\sqrt{k/m}\, \cos(\alpha \ell/2)$$

$\underline{\text{i.e.}}$ $V_{gr} = \ell\sqrt{k/m} \sim c_{s\,eff}$ for $\alpha \ell \lll 1$

(a la' sound)

but $\lim_{\alpha \to \pi/\ell} V_{gr} = \ell\sqrt{k/m}\,\cos(\pi/2) \to 0$

i.e. modes at edge of Brillouin zone non - propagating

modes in middle of zone propagate at acoustic speed.

Can also observe that:

$$x_{i+1} + x_{i-1} - 2x_i = e^{i[\alpha \ell i]}\left(e^{i\alpha\ell} + e^{-i\alpha\ell} - 2\right)$$

$$= 2e^{i[i\alpha\ell]}\left(\cos\alpha\ell - 1\right)$$

so $\cos\alpha\ell/1 \sim$ ratio of $(x_{i+1} + x_{i-1})/2x_i$

$\sim$ mean phase ratio

so $\alpha\ell \ll 1$ (in zone) $\Rightarrow$ neighbors on chain vibrate <u>in</u> phase $\to$ propagation
$\cos = 1$

$\alpha\ell \sim \pi$ (zone boundary) $\Rightarrow$ neighbors on chain vibrate <u>out</u> <u>of</u> phase $\to$ no propagation

$\cos = -1$

→ Boundary Conditions

Can distinguish 2 cases $\begin{cases} \text{periodic B.C.'s} \\ \text{fixed end B.C.'s.} \end{cases}$

1) Periodic B.C.'s

Now, $x_i = A e^{i[i\, l\alpha]}$

notational clarity ⇒ $x_n = A e^{i[n l \alpha]}$

$$1 < n < N.$$

For periodic B.C.'s,

$$x_n = x_{n+N} \Rightarrow e^{iNl\alpha} = 1$$

→ mode indx.

$$\therefore \quad N l \alpha = 2\pi p$$

$$\Rightarrow \boxed{\alpha = \frac{2\pi p}{N l}} \qquad p = \begin{cases} 0, \pm 1, \ldots \pm \frac{1}{2}(N-1) & N \text{ odd} \\ 0, \pm 1, \ldots \frac{1}{2} N & N \text{ even} \end{cases}$$

Note: guarantees N normal modes.

ii) Fixed end B.C.'s : $\left.\begin{aligned} X_0 &= 0 \\ X_{N+1} &= 0 \end{aligned}\right\}$ guarantees ends fixed

$$\Rightarrow \quad X_0 = X_{N+1} = 0$$

$$X_n = A e^{in\alpha\ell} + B e^{-in\alpha\ell}$$

$$= A\sin(n\ell\alpha) + B\cos(n\ell\alpha)$$

$B = 0 \rightarrow n = 0$ ✓

$(N+1)\alpha\ell = p\pi \quad ; \quad p = 1, \ldots N$

mode index

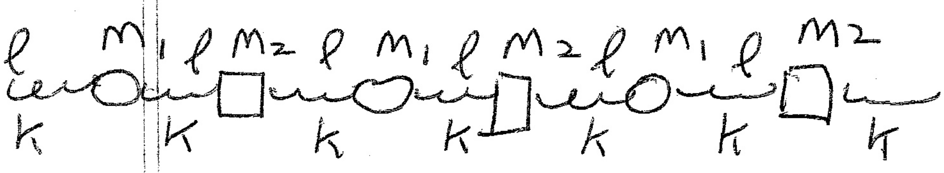
$$\Rightarrow \quad \boxed{\alpha_p = p\pi / \ell(N+1)}$$

$$\therefore \quad X_n(t) = A_n \sin\left(\frac{n\cancel{\ell}p\pi}{\cancel{\ell}(N+1)}\right) e^{-i\omega_p t}$$

where $\quad \omega_p^2 = \frac{4k}{m}\sin^2\left(\frac{p\pi\cancel{\ell}}{2\cancel{\ell}(N+1)}\right)$

– Diatomic Chain

→ consider slightly richer toy model, namely the diatomic chain



then, no loss of generality to associate

$$m_1 \rightarrow x_{2n}$$
$$m_2 \rightarrow x_{2n+1}$$

∴ can immediately write dynamical equations

$$m_1 \ddot{x}_{2n} = -k\left(2x_{2n} - x_{2n-1} - x_{2n+1}\right)$$

$$m_2 \ddot{x}_{2n+1} = -k\left(2x_{2n+1} - x_{2n} - x_{2n+2}\right)$$

solution of form:

$$x_{2n} = A e^{i2n\ell\alpha} e^{-i\omega t}$$

$$x_{2n+1} = B e^{i(2n+1)\ell\alpha} e^{-i\omega t}$$

$$-m_1\omega^2 A = -k\left(2A - (e^{il\alpha} + e^{-il\alpha})B\right)$$

$$-m_2\omega^2 B = -k\left(2B - (e^{il\alpha} + e^{-il\alpha})A\right)$$

$\Rightarrow$

$$(-m_1\omega^2 + 2k)A - k(2\cos l\alpha)B = 0$$

$$(-2k\cos l\alpha)A + (-m_2\omega^2 + 2k)B = 0$$

$\therefore$

$$\boxed{(\omega^2 - 2k/m_1)(\omega^2 - 2k/m_2) - \frac{4k^2}{m_1 m_2}\cos^2 l\alpha = 0}$$

$\Rightarrow$ dispersion relation:

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\left\{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4\sin^2(\alpha l)}{m_1 m_2}\right\}^{1/2}$$

$$1/\mu \equiv 1/m_1 + 1/m_2$$

$$\omega^2 = k/\mu \pm k/\mu\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}$$

$\therefore$ dispersion relation:

$$\boxed{\omega^2 = \frac{k}{\mu}\left\{1 \pm 1\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}\right\}}$$

can immediately observe:

→ system supports 2 modes

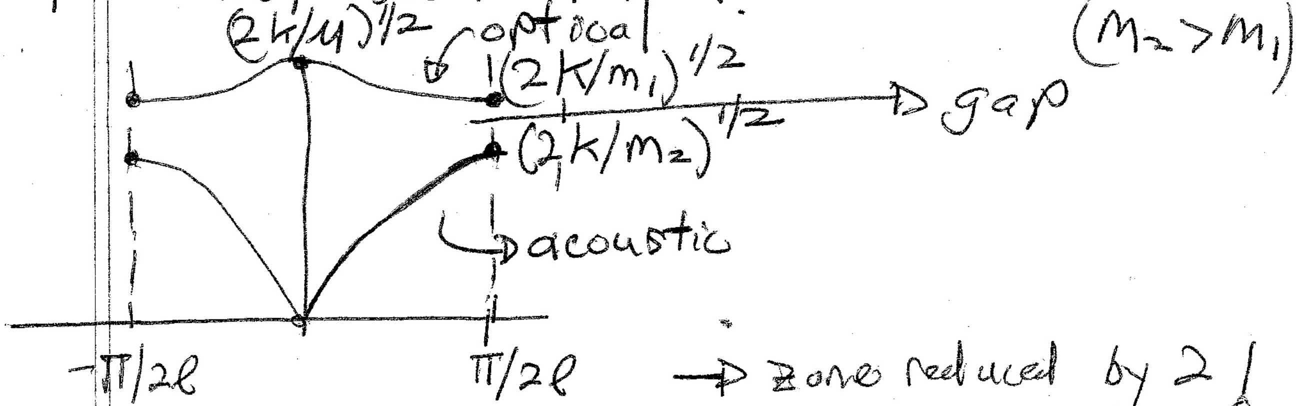
- low frequency → "acoustic" mode

→ analogous to mode of monatomic chain

- high frequency → "optical" mode (vibration)

→ new

Can plot dispersion relation.



→ zone reduced by 2!

Note: - acoustic mode $\omega \sim \alpha \left(\frac{k\ell^2}{m_2 + m_1} \right)^{1/2}$

$\sim k c_{s\,eff}$

as $k\ell \to 0$ ⇒ mass neighbors vibrate in phase

$x_n = x_{n+1}$

solid → phonon ($\omega = k c_s$)

optical mode $\omega \sim (2k/\mu)^{1/2}$

as $k\ell \to 0$; $m_1 x_n + m_2 x_{n+1} = 0$

i.e. neighboring masses vibrate <u>out</u> of phase, weighted by masses

solid $\to$ analogous collective mode is EM wave

$\omega^2 = \omega_{pe}^2 + c^2 k_1^2$ or plasmon

$\omega^2 = \omega_{pe}^2 + k^2 v_{Te}^2$

i.e. $k \to 0$, frequency constant!

$\to$ Note <u>gap</u> $\to$ no propagation for

$$(2k/m_2)^{1/2} < \omega < (2k/m_1)^{1/2}$$

$\to$ Consequence of fact

phonon $\leftrightarrow$ inertia of heavy mass

optical $\leftrightarrow$ inertia of light mass (in ω_p^2)

→ Transition to Continuum

To recover continuum $\left\{ \text{i.e.} \begin{array}{l} \text{elastic medium} \\ \text{massive string} \end{array} \right.$

take $\begin{array}{l} N \to \infty \\ \ell \to 0 \end{array}$ with constant $L = (N+1)\ell$

$$\left\{ \begin{array}{l} \frac{m}{\ell} = \mu = \text{const.} \\ K\ell = K = \text{const.} \end{array} \right.$$

Note: "$N \to \infty$" means $N > p$, for all modes p.

Then;

$$\omega_p^2 = \frac{4k}{m} \sin^2\left(\frac{p\pi}{2(N+1)}\right)$$

$$\cong \frac{4k}{m}\left(\frac{p\pi}{2(N+1)}\right)^2$$

$$= \frac{(p\pi)^2}{((N+1)\ell)^2} \frac{k\ell^2}{m}$$

$$= \left(\frac{p\pi}{L}\right)^2 \left(\frac{K}{\mu}\right) = \left(\frac{p\pi}{L}\right)^2 c_s^2$$

$$c_s^2 = k\ell^2/m = (k\ell)\,\ell/m = K/\mu$$

$$\to \omega^2 = k^2 c_s^2 \quad ; \quad \begin{array}{l} c_s^2 = K/\mu \\ k = p\pi/L \end{array}$$

f.) More Oscillations: Mechanics of Fields

→ Recall the string: (i.e. continuum limit)

$$\mathcal{L} = \mathcal{L}(y, y_t, y_x) \quad \to \text{Lagrangian density}$$

$$\mathcal{L} = \frac{1}{2}\mu y_t^2 - T\left[\left(1+(\partial y/\partial x)^2\right)^{1/2} - 1\right]$$

where $\mathcal{S} = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$

Then, for equation of motion

$$\delta\mathcal{S} = 0 \Rightarrow \delta\mathcal{S}$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\delta y_x \right)$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\frac{d}{dt}(\delta y) + \frac{\partial \mathcal{L}}{\partial y_x}\frac{d}{dx}(\delta y) \right)$$

$\delta y = 0$, at end pts

$$= \int_0^L dx \left.\frac{\partial \mathcal{L}}{\partial y_t}\delta y\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left.\frac{\partial \mathcal{L}}{\partial y_x}\delta y\right|_0^L$$

$\left.\begin{matrix}\delta y(x, t_2) \\ \delta y(x, t_1)\end{matrix}\right\} = 0$

traj. end points fixed $\delta q(t_2) = \delta q(t_1) = 0$

$$+ \int_{t_1}^{t_2} \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right)\delta y$$

Thus, have Lagrange equation of motion:

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \\ &\text{with b.c.:} \quad \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\Big|_0^L = 0 \end{aligned}}$$

clearly satisfied for free, fixed ends.

n.b. $\int_{t_1}^{t_2} dt\, \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\Big|_0^L$

[$\delta y(t, x) = 0$ only at t_2, t_1]

1) In 3D, $x_i = x, y, z$ for $i = 1, \ldots 3$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right)$$

2) For 1D string, small oscillation $\Rightarrow$

$$\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

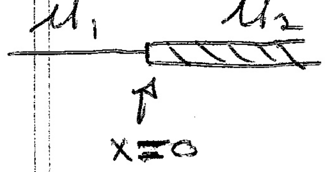
$$\frac{d}{dt}(\mu y_t) = -\frac{d}{dx}\left(\frac{-T y_x}{\sqrt{1+y_x^2}}\right)$$

$$\therefore \quad \mu y_{tt} = T\frac{\partial^2 y}{\partial x^2} \quad \to \text{wave equation}$$

3) Unambiguous formulation of basic equations for matching conditions

i.e. Consider 2 prototypical examples

i.)

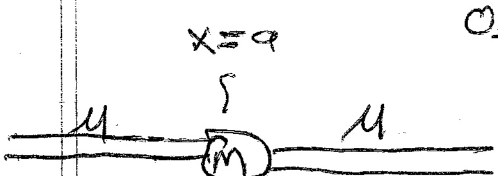


matching $\Rightarrow$ $y_-(0) = y_+(0)$

$$\int_{0_-}^{0_+} \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{\partial \mathcal{L}}{\partial y} + \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right)\right\} = 0$$

so $$\left.\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right|_{0_+} = \left.\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right|_{0_-}$$

ii.)



(continuity understood)

$$\mu = \mu + M\delta(x-a)$$

$$\mathcal{L} = \frac{1}{2}\left(\mu + M\delta(x-a)\right) y_t^2 - \frac{T}{2}(y_x)^2$$

$$\left(\mu + M\delta(x-a)\right) y_{tt} = T y_{xx}$$

$$y = \hat{y}(x)\, e^{-i\omega t}$$

$$T\hat{y}_{xx} = -\omega^2\left(\mu + M\delta(x-a)\right)\hat{y}$$

$$\int_{a_-}^{a_+} \left[T\hat{y}_{xx} + \omega^2\left(\mu + M\delta(x-a)\right)\hat{y}\right] = 0$$

$$T\,\hat{y}_x\Big|_{a_-}^{a_+} = -\omega^2 M\,\hat{y}(a)$$ $\Rightarrow$ jump condition.

N.B.: Use of Lagrangian ab-initio renders all questions re: order of derivatives moot.

→ Hamiltonian Fomulation

As usual, can define canonical momentum

$$\pi = \partial\mathcal{L}/\partial y_t$$

N.B. $\pi = \mu y_t$, for string.

Similarly, can define Hamiltonian density:

$$\mathcal{H} = \pi\dot{y} - \mathcal{L}$$

for $\partial\mathcal{L}/\partial t = 0$, $\underbrace{\mathcal{H} = \mathcal{E}}_{\text{energy density}}$

and Hamiltonian $H = \int_0^L dx\,\mathcal{H}$.

For string:

$$\mathcal{H} = \frac{\pi^2}{\mu} - \mathcal{L} = \underbrace{\frac{\pi^2}{2\mu}}_{KE} + \underbrace{\frac{T(y_x)^2}{2}}_{Pot.\,E.}$$

Hamilton's equations then follow from Principle of Least Action, i.e.

$$S = \int_{t_1}^{t_2} dt \int_0^L dx \,(\pi y_t - \mathcal{H})$$

$$\begin{cases} \mathcal{L} = \pi y_t - \mathcal{H} \\ \mathcal{H} = \mathcal{H}(\pi, y_x, y) \\ \mathcal{L} = \mathcal{L}(y_t, y_x, y) \end{cases}$$

so

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left(y_t \,\delta\pi + \pi\,\delta y_t - \left(\frac{\partial \mathcal{H}}{\partial \pi}\delta\pi + \frac{\partial \mathcal{H}}{\partial y_x}\delta y_x + \frac{\partial \mathcal{H}}{\partial y}\delta y \right)\right)$$

ignoring surface terms

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ y_t\,\delta\pi - \left(\frac{\partial}{\partial t}\pi\right)\delta y - \frac{\partial \mathcal{H}}{\partial \pi}\delta\pi - \left(\frac{\partial \mathcal{H}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{H}}{\partial y_x}\right)\right)\delta y \right\}$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \left(y_t - \frac{\partial \mathcal{H}}{\partial \pi}\right)\delta\pi + \delta y\left(\frac{\partial \mathcal{H}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{H}}{\partial y_x}\right) + \frac{\partial \pi}{\partial t}\right)\right\}$$

$\delta S = 0 \Rightarrow$

$$\begin{cases} \dot{y} = \dfrac{\partial \mathcal{H}}{\partial \pi} \\[2ex] \dot{\pi} = -\dfrac{\partial \mathcal{H}}{\partial y} + \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{H}}{\partial y'}\right) \end{cases}$$

Now, can observe furthur; $\partial_t \mathcal{L} = 0$

$$\mathcal{H} = \pi \dot{y} - \mathcal{L}$$

so $$\frac{d}{dt}\mathcal{H} = \pi \ddot{y} + \dot{\pi}\dot{y} - \frac{d\mathcal{L}}{dt}$$

$$= \cancel{\pi \ddot{y}} + \dot{\pi}\dot{y} - \left(\frac{\partial \mathcal{L}}{\partial y}\dot{y} + \cancel{\frac{\partial\mathcal{L}}{\partial \dot{y}}\ddot{y}} + \left(\frac{\partial\mathcal{L}}{\partial y_x}\right)\dot{y}_x\right)$$

$$(\pi = \partial\mathcal{L}/\partial\dot{y})$$

$$\frac{d\mathcal{H}}{dt} = \dot{\pi}\dot{y} - \left(\frac{\partial\mathcal{L}}{\partial y}\dot{y} + \frac{\partial\mathcal{L}}{\partial y_x}\dot{y}_x\right)$$

but furthur $$\frac{\partial\mathcal{L}}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial y_x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

$$\frac{d\mathcal{H}}{dt} = \cancel{\dot{\pi}\dot{y}} - \dot{y}\left(\cancel{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)\right) - \frac{\partial\mathcal{L}}{\partial y_x}\dot{y}_x$$

as $\pi = \partial\mathcal{L}/\partial\dot{y}$

$$\Rightarrow \frac{d\mathcal{H}}{dt} = -\dot{y}\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial y_x}\right) - \left(\frac{\partial\mathcal{L}}{\partial y_x}\right)\frac{\partial}{\partial x}(\dot{y})$$

$$= -\frac{\partial}{\partial x}\left(\dot{y}\,\partial\mathcal{L}/\partial y_x\right)$$

Thus, have shown (in general) $\Rightarrow$

$$\frac{d\mathcal{H}}{dt} + \frac{\partial}{\partial x}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0$$

and can generalize to higher dimensions

$$\boxed{\frac{d\mathcal{H}}{dt} + \sum_i \frac{\partial}{\partial x_i}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right) = 0}$$

What does it mean ?

Here $\mathcal{H} = \mathcal{E} \equiv$ energy density

so relation of form :

$$\frac{d\mathcal{H}}{dt} + \underline{\nabla} \cdot \underline{S} = 0 \quad ; \quad S_i = \dot{y}\frac{\partial \mathcal{L}}{\partial y_{x_i}}$$

Poynting theorem !, with :

$S_x = \dot{y}\dfrac{\partial \mathcal{L}}{\partial y_x}$ as $\left\{\begin{array}{l}\text{Poynting flux} \\ \text{i.e. wave energy density flux in} \\ \text{direction of wave propagation}\end{array}\right.$

For string

$$S_x = \dot{y}\,\partial \mathcal{L}/\partial y_x = -T\dot{y}\,y_x$$

Note:

→ Poynting thm. relates (local) wave energy density with wave energy density flux, i.e.

$$\frac{d\mathcal{H}}{dt} + \partial_x S_x = 0$$

→ Poynting thm. relates rate of energy change to wave energy density flux thru interval

i.e.

$$\frac{d}{dt} E = \frac{d}{dt} \int_{x_1}^{x_2} \mathcal{H}\, dx = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} S_x$$

$$= -S_x \Big|_{x_1}^{x_2}$$

→ Poynting thm. formed by expressing $\frac{d}{dt}\mathcal{E}$ as $\underline{\nabla} \cdot \underline{S}$, etc.

recall in E and M:

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \times \underline{E} = -\frac{1}{c}\, \partial \underline{B} / \partial t$$

but $\mathcal{E} = E^2/8\pi + B^2/8\pi$

then $\left(\frac{\partial \underline{E}}{\partial t} = c \underline{\nabla} \times \underline{B} - 4\pi \underline{J} \right) * \cdot \underline{E}/4\pi$

$\left(\frac{\partial \underline{B}}{\partial t} = - c \underline{\nabla} \times \underline{E} \right) \cdot \left(\underline{B}/4\pi \right)$

$\Rightarrow$

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) = - \underline{E} \cdot \underline{J} - \underline{\nabla} \cdot \underbrace{\left(\frac{c}{4\pi} \underline{E} \times \underline{B} \right)}_{\underline{S}}$$

i.e. form Poynting thm. by considering time rate of change of energy density.

$\rightarrow$ Important to distinguish:

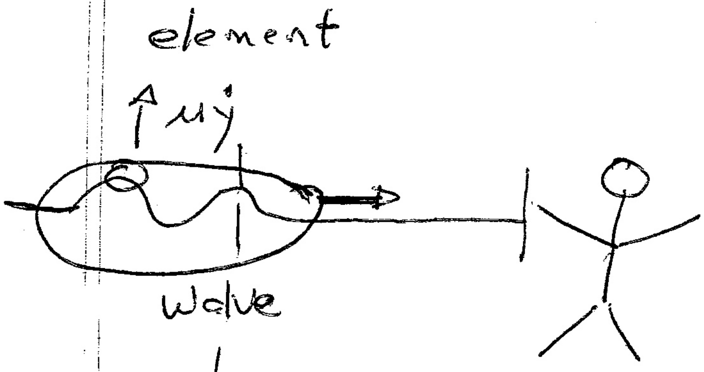
$\underline{\Pi} = \mu \dot{y} \hat{y} \equiv$ canonical momentum

$\rightarrow$ momentum of string element $\mu \dot{y}(x,t)$, in $\hat{y}$ direction

$$\underline{S} = - T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \hat{x} = \frac{\partial \mathcal{L}}{\partial y_x} \frac{\partial y}{\partial t} \hat{x}$$

$\equiv$ wave energy density flux

$\rightarrow$ momentum of wave / fluctuation, in $\hat{x}$ direction i.e.



→ "feels" wave momentum (kick in $\vec{x}$)

flux thru (x) at time t $= S_x(x,t)$

calculating for wave on string:

if $y = A\cos(k(x - v_{ph}t))$ $\qquad v_{ph} = (T/\mu)^{1/2}$

$$\frac{\partial y}{\partial t} = +A k v_{ph} \sin(k(x - v_{ph}t))$$

$$\frac{\partial y}{\partial x} = -A k \sin(k(x - v_{ph}t))$$

$$S_x = +T A^2 k^2 v_{ph} \sin^2(k(x - v_{ph}t))$$

$$\therefore \quad \overline{S}_x = \frac{T k^2 v_{ph}}{2} A^2$$

but: $\omega^2 = v_{ph}^2 k^2$

$$\boxed{\overline{S}_x = \frac{\mu \omega^2 v_{ph}}{2} A^2}$$

→ Wave Momentum

- have developed notions of wave energy and Poynting Theorem, ∴

- natural to investigate wave momentum.

Now, recall in E+M,

$$\underline{P}_{EM} = \frac{1}{c^2}\,\underline{S} = \frac{1}{4\pi c}\,\underline{E}\times\underline{B}$$

↓ momentum of electromagnetic wave

↳ Poynting vector

Thus, natural motivation to investigate relation for string, i.e.

$$S = \dot{y}\,\frac{\partial \mathcal{L}}{\partial y_x}$$

so

$$\dot{S} = \ddot{y}\,\frac{\partial \mathcal{L}}{\partial y_x} + \dot{y}\,\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

for string;

$$\ddot{y} = \frac{T}{\mu}\,y_{xx} = V_{ph}^2\,y_{xx} \quad ; \quad \frac{\partial \mathcal{L}}{\partial y_x} = -T y_x$$

$$\dot{S} = \left\{ -\frac{T}{\mu} y_{xx} T y_x - \mu \dot{y} \frac{T}{\mu} \frac{d}{dt} y_x \right\}$$

$$= -\frac{T}{\mu} \frac{\partial}{\partial x} \left\{ \frac{T y_x^2}{2} + \frac{\mu \dot{y}^2}{2} \right\}$$

$$= -V_{ph}^2 \frac{\partial}{\partial x} \mathcal{E}$$

$\therefore$ if define wave momentum density $\underline{P}_w = \frac{1}{V_{ph}^2} \underline{S}$,

have natural conservation law

$$\boxed{\frac{d}{dt} \underline{P}_w + \underline{\nabla} \mathcal{H} = 0}$$

here $\underline{\nabla} \mathcal{H} = \underline{\nabla} \mathcal{E}$ is force density. Then wave stress (pushes in direction of propagation)

for $\underline{P} = \int_{x_1}^{x_2} dx\, \underline{P}_w$ in a chunk of string,

wave momentum

$$\frac{d}{dt} P + \mathcal{H} \Big|_{x_1}^{x_2} = 0$$

difference/jump in energy across chunk.

Thus, have complete energy, momentum relations.

$$\boxed{\begin{aligned} &\frac{d}{dt}\mathcal{H} + \underline{\nabla}\cdot\underline{S} = 0 && \underline{S} = \dot{y}\,\frac{\partial \mathcal{L}}{\partial y_x}\,\hat{c}_w \\ &\frac{\partial}{\partial t}\underline{P_w} + \underline{\nabla}\mathcal{H} = 0 && \underline{P_w} = \frac{1}{v_{ph}^2}\,\underline{S}\,\hat{c}_w \end{aligned}}$$

→ Can derive from divergence relation for stress tensor (E+M).

An application: Sound

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{V}) = 0 \qquad P = P(\rho)$$

$$\frac{dP}{d\rho} \equiv c_s^2$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{1}{\rho}\,\underline{\nabla} P$$

linearizing ⇒

$$\frac{\partial \hat{\underline{V}}}{\partial t} = -\frac{1}{\rho_0}\,c_s^2\,\underline{\nabla}\hat{\rho}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -\rho_0\,\underline{\nabla}\cdot\hat{\underline{V}}$$

Note: Can write:

$$\frac{\partial \mathcal{H}}{\partial t} + \underline{\nabla}\cdot\underline{S} = 0$$

$$\frac{\partial}{\partial t}\underline{P}_w + \underline{\nabla}\mathcal{H} = 0$$

so

$$\begin{pmatrix} \frac{1}{v_{ph}}\partial/\partial t \\ \partial/\partial x \end{pmatrix} \begin{bmatrix} \mathcal{H} & S/v_{ph} \\ S^*/v_{ph} & \mathcal{E} \end{bmatrix} = 0$$

$\mathcal{E} = \mathcal{H}$, here.

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu} \equiv$ energy-momentum tensor of string

$$\partial_\mu = \left(\frac{1}{v_{ph}}\partial_t,\ \partial_x\right)$$

.. E+M:

$$T^{ik} = \begin{pmatrix} W & S_x/c & S_y/c & S_z/c \\ S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left\{-E_\alpha E_\beta - H_\alpha H_\beta + \frac{1}{2}\delta_{\alpha\beta}(E^2+H^2)\right\}$$

$\downarrow$

Maxwell stress tensor.

then: $$\frac{\partial^2 \hat{\rho}}{\partial t^2} = C_s^2 \nabla^2 \hat{\rho} = \rho_0 \nabla \cdot \left\{ \frac{C_s^2}{\rho_0} \nabla \rho \right\}$$

For energy-momentum relations:

$$(1) \cdot \hat{V} \rho_0 + (2) \cdot \frac{\hat{\rho} C_s^2}{\rho_0} \Rightarrow$$

$$\frac{\partial}{\partial t} \left(\frac{\rho_0 \hat{V}^2}{2} \right) + C_s^2 \hat{V} \cdot \nabla \hat{\rho} = 0$$

$$\frac{\partial}{\partial t} \left(\frac{\hat{\rho}^2 C_s^2}{2 \rho_0} \right) + C_s^2 \hat{\rho} \, \nabla \cdot \hat{V} = 0$$

$$\therefore \frac{\partial}{\partial t} \left(\frac{\rho_0}{2} \hat{V}^2 + \frac{\hat{\rho}^2 C_s^2}{2 \rho_0} \right) + \nabla \cdot \underbrace{\left[C_s^2 \rho V \right]}_{S} = 0$$

$$\mathcal{H} = \mathcal{E} = \underbrace{\frac{\rho_0 \hat{V}^2}{2}}_{T} + \underbrace{\frac{\hat{\rho}^2 C_s^2}{2 \rho_0}}_{V}$$

Similarly,

$$P_w = \frac{1}{C_s^2} S$$

$$\frac{\partial P_w}{\partial t} = \frac{\partial}{\partial t} (\rho V) = \frac{\partial \hat{\rho}}{\partial t} \hat{V} + \hat{\rho} \frac{\partial V}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \hat{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{\hat{V}}$$

$$\frac{\partial \underline{\hat{V}}}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \hat{\rho}$$

$$\frac{\partial P_w}{\partial t} = -\rho_0 \frac{\underline{\nabla}}{2} \cdot [\underline{V}\,\underline{V}] - \frac{c_s^3}{\rho_0} \underline{\nabla} \cdot \left(\frac{\hat{\rho}^3}{2}\right)$$

$$= -\underline{\nabla}\left(\frac{\rho V^2}{2} + \frac{c_s^2}{\rho_0}\frac{\hat{\rho}^3}{2}\right) \quad \checkmark$$

for longitudinal waves.