Canonical Perturbation Theory:

Secular P.T., Tori, Island Formation

Canonical Perturbation Theory – An Introduction

→ not surprisingly, few systems are integrable, in sense of action-angle variables, i.e.

can be written as: $H = H(I_1, \dots I_n)$

$\dot{I}_i = 0 \quad \forall i$ $\qquad\qquad \dot{\theta}_i = \frac{\partial H}{\partial I_i}$

$\frac{\partial H}{\partial \theta_i} = 0$ $\qquad\qquad \theta_i = \omega(I_i)\, t + \theta_{i_0}$

→ thus, frequently encounter/view the system as perturbation about integrable system

i.e. for 1 degree of freedom:

$$H = \underbrace{H_0(I)}_{\text{unperturbed, integrable motion}} + \underbrace{\epsilon H_1(I, \theta)}_{\text{perturbation} \leftrightarrow \text{breaks symmetry}}$$

→ seek to "integrate" H perturbatively, where "integrate" ⇒

Canonically transform $\underset{\text{(old)}}{I, \theta} \to \underset{\text{(new)}}{J, \phi}$

such that: $\left.\begin{aligned} \dot{J} &= 0 \\ \dot{\phi} &= \omega(J) \end{aligned}\right\}$ to specified order in P.T.

obviously understand here that:

$$J = I + O(\epsilon)$$
$$\phi = \theta + O(\epsilon) \qquad \text{etc}$$

→ Proceeding:

- note structure:

old: I, θ

new: J, ϕ

s/t $\dot{J} = 0$, to $O(\epsilon)$

{known property of new Hamiltonian}

⇒ type 2, a la' H-J, transformation:

$p \leftrightarrow I$, $P \leftrightarrow J$
$q \leftrightarrow \theta$, $Q \leftrightarrow \phi$

;

and	indep	dep
	$q \leftrightarrow \theta$	$p \leftrightarrow I$
	$P \leftrightarrow J$	$Q \leftrightarrow \phi$

so, can write

$$p = \frac{\partial F}{\partial q} = \frac{\partial S}{\partial q} \quad \Rightarrow \quad I = \frac{\partial S}{\partial \theta}$$

($F = S$, here)

$$Q = \frac{\partial F}{\partial p} = \frac{\partial S}{\partial p} \Rightarrow \phi = \frac{\partial S}{\partial J}$$

But here:

$$S = S_0 + \epsilon S_1 + \ldots\ldots$$

$$\underbrace{H'(J)}_{\substack{\text{new Hamiltonian} \\ \text{(fctn } J \text{ only)}}} \equiv \underbrace{K(J)}_{\text{(re-label)}} = K_0(J) + \epsilon K_1(J) + \ldots$$

$$\Rightarrow K(J) \equiv H(I, \theta)$$
$$\equiv H_0\left(\frac{\partial S}{\partial \theta}, \theta\right) + \epsilon H_1\left(\frac{\partial S}{\partial \theta}, \theta\right) + \ldots$$

Now here, $S = S_0 + \epsilon S_1$
$J = I + O(\epsilon)$
$\phi = \theta + O(\epsilon)$

$\theta \rightarrow$ old q

$$\therefore \quad S_0 = J\theta \qquad \left(\begin{array}{c} I = J \quad \phi = \theta \\ \text{to lowest order} \end{array}\right)$$

$J \leftarrow$ new p

so $\quad S = J\theta + \epsilon S_1 + \ldots$

- Now, plugging S into $H' = K = H$ equation:

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J) + \ldots$$

$$= H_0\left(J + \epsilon \frac{\partial S_1}{\partial \theta} + \epsilon^2 \frac{\partial S_2}{\partial \theta} + \ldots\right)$$

$$+ \epsilon H_1\left(J + \epsilon \frac{\partial S_1}{\partial \theta} + \ldots, \theta\right)$$

grinding it out:
to $O(\epsilon^2)$

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J) = H_0(J) + \epsilon \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J}$$

$$+ \epsilon^2 \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J} + \epsilon H_1(J, \theta) + \epsilon^2 \frac{\partial S_1}{\partial \theta} \frac{\partial H_1}{\partial J}(J)$$

$$+ \frac{1}{2} \epsilon^2 \left(\frac{\partial S_1}{\partial \theta}\right)^2 \frac{\partial^2 H_0}{\partial J^2}$$

matching term-by-term:

$$H_0 = K_0$$

$$K_1(J) = \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + H_1(J, \theta)$$

if

$$K_2(J) = \frac{1}{2}\left(\frac{\partial S_1}{\partial \theta}\right)^2 \frac{\partial^2 H_0}{\partial J^2} + \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J} + \frac{\partial S_1}{\partial \theta} \frac{\partial H_1}{\partial J} + H_2$$

etc.

For $\mathcal{O}(\epsilon)$:

$$K_1(J) = \frac{\partial S_1}{\partial \theta}\frac{\partial H_0}{\partial J} + H_1(J,\theta)$$

$$= \frac{\partial S_1}{\partial \theta}\omega_0(J) + H_1(J,\theta)$$

where understand : $J = \partial S/\partial \theta$

$$= I + \epsilon \frac{\partial S_1}{\partial \theta}$$

and $\phi = \frac{\partial S}{\partial J} = \theta + \epsilon \frac{\partial S_1}{\partial J}$

$$\therefore \left\{ \boxed{\begin{aligned} \theta &= \phi - \epsilon\, \partial S_1/\partial J \\ I &= J + \epsilon\, \partial S_1/\partial \theta \end{aligned}} \right.$$

Now, if define:

$$H_1 = \langle H_1 \rangle + \tilde{H}_1$$

$$\langle H_1 \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} H_1$$

can average :

$$\Rightarrow \quad \boxed{K_1(J) = \langle H_1 \rangle}$$

but need $S_1 \Longleftrightarrow$ obtain from solvability

$$\omega_0(J) \frac{\partial S_1}{\partial \theta} = K_1(J) - H_1$$

$$= K_1(J) - \langle H_1 \rangle - \tilde{H}_1$$

$$\therefore \quad \boxed{\omega_0(J) \frac{\partial S_1}{\partial \theta} = -\tilde{H}_1}$$

Now, $\tilde{H}_1 = \sum_{n=1}^{\infty} H_n(J)\, e^{in\theta}$

$$S_1 = \sum_{n=1}^{\infty} S_n\, e^{in\theta}$$

so

$$S = J\theta + S_1$$

$$S_1 = \sum_n - \frac{H_n(J)}{in\omega_0(J)} e^{in\theta}$$

so finally can write full solution to $O(\epsilon)$:

$$\boxed{\begin{aligned} \phi &= \theta + \epsilon \frac{\partial S_1}{\partial J}(J,\theta) \\ J &= I - \epsilon \frac{\partial S_1}{\partial \theta}(J,\theta) \\ \omega &= \omega_0(J) + \epsilon \frac{\partial}{\partial J} K_1(J) \end{aligned}}$$

where:
$$\boxed{\begin{aligned} K_1 &= \langle H_1 \rangle_\theta \\ S_1 &= \sum_n \frac{i n H_n(J)}{n\omega_0(J)} e^{in\theta} \end{aligned}}$$

Now:

- if multi-degrees freedom

$\theta \rightarrow \underline{\theta}$

$n\omega_0(J) = \underline{n} \cdot \underline{\omega}_0(\underline{J})$

$\therefore$ "small denominators" (at / near) resonant tori!

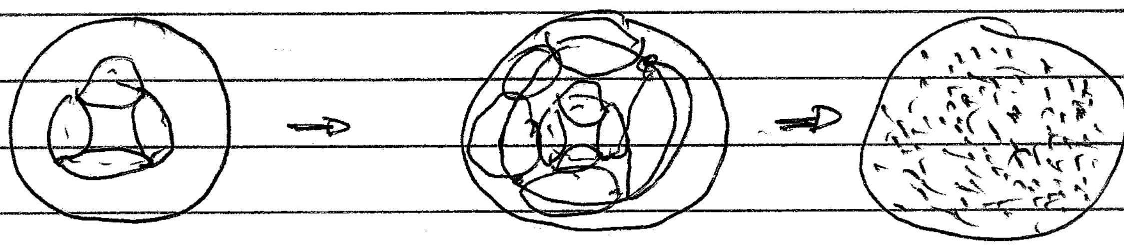
$\Rightarrow$ what happens to tori?

$\Rightarrow$ presents strong question re: convergence of perturbation expansion.

– Resolution:

→ KAM theorem: 'Most' surfaces preserved. Destruction limited to $O(\epsilon)$ volume.

→ chaos 'seeded' at resonant/rational tori', where islands form. Island overlap ⇒ volume filling chaos

i.e. 

Some examples:

i.) Trivial → Pendulum (yet again)

For simple pendulum,

$$H_p = \tfrac{1}{2} G p^2 - F\cos\phi = E$$

expanding: $H_p = \tfrac{1}{2} G p^2 - \cancel{F} + \tfrac{1}{2} F\phi^2 - \frac{F\phi^4}{4!}$ (irrel.) ... next order term

can take $\epsilon \to 1$ at end.

$\therefore$ $H = H_0 + \epsilon H_1$

$$H_0 = \tfrac{1}{2} G p^2 + \tfrac{1}{2} F\phi^2$$

$$H_1 = -\frac{F\phi^4}{4!}$$

1) Canonical Form

Know: $H_0 = I\omega_0$ $\quad (I = E/\omega)$

use canonical transformation:

$$q = (2I/R)^{1/2} \sin\theta$$

$$R = (F/G)^{1/2}$$

$$p = (2IR)^{1/2} \cos\theta$$

$$\omega_0 = (FG)^{1/2}$$

$$H = \omega_0 I - \frac{\epsilon}{6} G I^2 \sin^4\theta$$

$$H_1 = -\frac{G I^2}{48}\left(3 - 4\cos 2\theta + \cos 4\theta\right)$$

seek $\begin{cases} \text{- correction to Hamiltonian and frequency} \\ \text{- generating function} \end{cases}$

For new Hamiltonian $\bar{H}$:

$$\bar{H} = H_0(\bar{J}) + \epsilon\left\langle H_1(\bar{J}, \bar{\theta})\right\rangle$$

$$\langle\ \rangle = \frac{1}{2\pi}\int_0^{2\pi}$$

$\therefore$

$$\boxed{\begin{aligned} \bar{H} &= \omega_0 \bar{J} - \frac{\epsilon G \bar{J}^2}{16} \\ \omega &= \frac{\partial \bar{H}}{\partial \bar{J}} = \omega_0 - \frac{\epsilon G \bar{J}}{8} \end{aligned}}$$

, upon averaging

N.B. Perturbation lowers freq.

and $\omega \dfrac{\partial S_1}{\partial \bar{\theta}} = -\left(H_1 - \langle H_1 \rangle\right)$

$$\Rightarrow \boxed{S_1 = -\frac{G J^2}{192\,\omega_0}\left(8\sin 2\theta - \sin 4\theta\right)}$$

ii.) *Explicit Time Dependence*

i.e. Consider : $H = H_0(I) + \epsilon\, H_1(I, \theta, t)$

time

$\therefore \rightarrow H_1 = \sum_{\ell, m} H_{1_{\ell,m}}(I)\, e^{i(\ell\theta + m\Omega t)}$

{expand in space
time harmonics.

$\rightarrow$ now must use C.T.
expression:

$$\bar{H} = H + \frac{\partial F}{\partial t}$$

$$= H + \epsilon \frac{\partial S_1}{\partial t}$$

{time derivative of generating function enters transformation of H

Now, $S = J\theta + \epsilon S_1(J, \theta, t)$

$$\bar{H}(J, \bar{\theta}, t) = H(I, \theta, t) + \epsilon \frac{\partial S_1}{\partial t}$$

$$\Rightarrow \quad \bar{H}_0 = H_0(J)$$

$$\bar{H}_1 = \frac{\partial S_1}{\partial t} + \omega \frac{\partial S_1}{\partial \theta} + H_1 \qquad (\epsilon \text{ out})$$

– Now must choose S_1 to <u>eliminate</u> oscillating part of H_1, i.e.

→ to eliminate, need average over space and time

i.e.

$$\bar{H} = H_0 + \epsilon \langle H_1 \rangle_{\theta, t} \, , \quad \boxed{\omega = \omega_0 + \epsilon \frac{\partial}{\partial J}\langle H_1 \rangle}$$

$$\frac{\partial S_1}{\partial t} + \omega \frac{\partial S_1}{\partial \theta} = -\tilde{H}_1 \qquad (\epsilon \text{ out})$$

so

$$\boxed{S_1 = i \sum_{\ell, m \neq 0} \frac{H_{1\,\ell m}}{\ell \omega(J) + m\Omega} \exp\left[i(\ell\theta + m\Omega t)\right]}$$

– a la' 2 deg. freedom, have potentially singular denominator

i.e. $\underline{m} \cdot \underline{\omega} = 0$

here, $\ell\omega + m\Omega = 0 \Rightarrow m = -1$

$(H_1 \sim E)$ wave – particle resonance!

→ General Properties of Motion in s dimensions.

Now, consider: system
- s degrees of freedom (arbitrary)
- separable H.J. equation

$$S = \sum_{i=1}^{s} S_i(\xi) \qquad \text{(i.e. integrable)}$$

∴ can define $\underline{\underline{s}}$ action variables I_i

$$I_i = \oint \frac{p_i dq_i}{(2\pi)}$$

and $\theta_i = \partial S_0 / \partial I_i$ angle variables

so

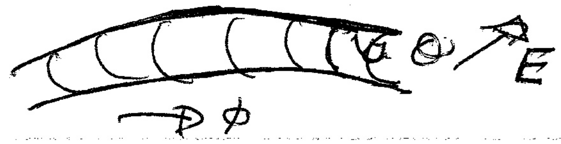
$$\dot{I}_i = 0$$

$$\theta_i = \omega_i(E)t + t_0$$
$$\omega_i(E) = \partial E / \partial I_i$$

i.e. for $s = 2$

$$\dot{I}_1 = \dot{I}_2 = 0 \qquad \omega_{\substack{1\\2}} = \partial E / \partial I_{\substack{1\\2}}$$

$$\theta_{\substack{1\\2}} = \omega_{\substack{1\\2}}(E)t + t_0$$



∴ phase space is 2 torus. Fixed $E \Rightarrow$ motion on toroidal surface.

[In general, phase space is s-torus.]

$$\theta = \omega_1(E)\, t \qquad\qquad \theta = \frac{\omega_1(E)}{\omega_2(E)}\,\phi$$
$$\phi = \omega_2(E)\, t$$

→ now, for any $F(\underline{q}, \underline{p})$, can write:

$$F = \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_s} A_{\ell_1, \ell_2 \ldots s} \exp\left[i(\ell_1 \theta_1 + \ell_2 \theta_2 + \cdots \ell_s \theta_s) \right]$$

$\ell_1, \ell_2, \ldots \ell_s$ integers.

equivalently:

$$F = \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_s} A_{\ell_1, \ell_2 \ldots \ell_s} \exp\left[i t \left(\underline{\ell} \cdot \underbrace{\frac{\partial E}{\partial \underline{I}}}_{\omega_i} \right) \right]$$

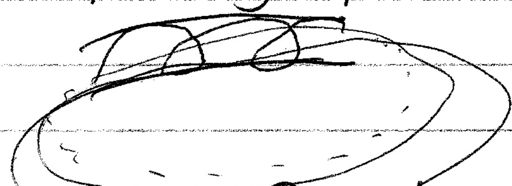
$$\underline{\ell} \cdot \frac{\partial E}{\partial \underline{I}} = \ell_1 \frac{\partial E}{\partial I_1} + \ell_2 \frac{\partial E}{\partial I_2} + \cdots \ell_s \frac{\partial E}{\partial I_s}.$$

Now, in general:

→ frequencies not commensurate, so F not periodic i.e. $\underline{l} \cdot \partial E / \partial \underline{I}$ irrational

→ indeed, system generally not periodic in any coordinate (except for special E).

but, for sufficient time, system will come arbitrarily close to starting point. → [Poincare Recurrence Thm.]



sufficient windings.

∴ motion is "conditionally" periodic.

But; degeneracy happens!

- degeneracy $n\omega_i = m\omega_j$
- all commensurate ⇒ complete degeneracy.

So, as in Kepler problem, ⇒ degeneracy implies reduction in number of independent I_i. Why?

Commensurate frequencies $\Rightarrow$

$$n_1 \omega_1 = n_2 \omega_2$$

$$n_1 \frac{\partial E}{\partial I_1} = n_2 \frac{\partial E}{\partial I_2}$$

so $E = E(n_2 I_1 + n_1 I_2)$

i.e. - energy depends on sum of action variables

$\Rightarrow$

- degeneracy

$\Rightarrow$

- can make canonical transformation so $E = E(I')$, only.

$\Rightarrow$ $\therefore$ in degenerate motion, there is an increase in the number of one-valued integrals of the motion, relative to non-degenerate case.

i.e. non-degenerate motion - s degs freedom

$2s-1 \rightarrow$ IOM'S

$\left\{ \begin{array}{l} s \text{ values } I_i \Rightarrow \text{ single valued } I_i \\ s-1 \text{ values a la } \Theta_i \, \partial E/\partial I_k - \Theta_k \, \partial E/\partial I_i \end{array} \right.$

note: S−1 values → phases (i.c.'s) of angle variables.

→ not single valued.

but if degeneracy, note though

→ $n_1\theta_1 - n_2\theta_2$ not single valued

(it is, to addition of 2π)

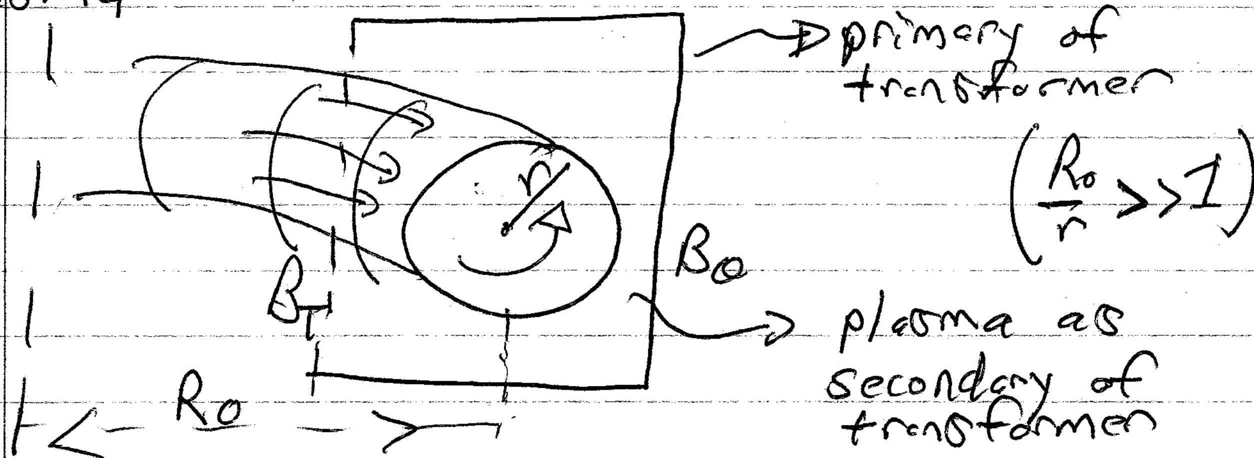
so

→ $\sin(n_1\theta_1 - n_2\theta_2)$ is single valued,
(etc.)

(Case Study)
Aside: Magnetic Field Lines on a Tokamak: A Practical Example of Phase Space Evolution on Tori

→ What is a Tokamak?

- toroidal confinement device for magnetized plasma



$I = I_0 \Rightarrow B_T(R) = 2I_0/cR$ → toroidal field (external)

$B_T \gg B_\theta$ $\quad B_\theta(r) = \int_0^r r'dr' \frac{J_T}{c}(r')$ → poloidal field (plasma current)

- $B_\theta(r)$ → shorts out charge separation due ∇B drift
  → stability, confinement
  → heating (ohmic)

more info: "Tokamak Plasma, A Complex Physical System" B.B. Kadomtsev

Minimal Model: The "Toroidal Cow" for Toroidal Field Configurations

- tokamak as periodic cylinder with:

$$\left.\begin{array}{l} 0 < r < a \\ L = 2\pi R_0 \end{array}\right\} \left\{\begin{array}{l} B_z = B_T \quad \text{(uniform, external)} \\ B_\theta(r) \end{array}\right.$$

$$\langle \underline{B} \rangle = B_\theta(r)\,\hat{\theta} + B_z \hat{z}$$

$$\underline{B} = \langle \underline{B} \rangle + \underline{\tilde{B}}_\perp$$

- field line:

$$\frac{dz}{B_T} = \frac{r\,d\theta}{B_\theta(r) + \tilde{B}_\theta} = \frac{dr}{\tilde{B}_r}$$

$\therefore$

$$\frac{d\theta}{dz} = \frac{1}{r}\frac{B_\theta(r) + \tilde{B}_\theta}{B_z} \cong \frac{1}{r}\frac{B_\theta(r)}{B_z} \qquad \tilde{B}_\theta \ll B_\theta(r)$$

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_z}$$

For un-perturbed field configuration:

$$\frac{d\theta}{dz} = \frac{1}{r}\frac{B_\theta(r)}{B_z} = \frac{1}{R\,q(r)} \, ; \qquad q(r) \equiv B_z r / R B_\theta(r)$$

$q(r)$ → safety factor

$$\frac{dr}{dz} = 0 \quad \text{(no radial wandering)}$$

$\iota(r) \equiv$ winding rate (i.e. rotational transform)
(# poloidal circuits per toroidal)

→ Relation to Hamiltonian Dynamics ?

$\frac{dx}{dz} = \frac{\tilde{B}_x}{B_0}, \quad \frac{dy}{dz} = \frac{B_y(x) + \tilde{B}}{B_0} \quad \nabla \cdot \tilde{B} = 0 \Rightarrow$ Ham.

Useful to observe similarity between:

a) Hamiltonian System with:

$$H = H(x,y) \quad \text{so} \quad \begin{cases} \dot{x} = -\partial H/\partial y \\ \dot{y} = \partial H/\partial x \end{cases} \qquad (\underline{\nabla} \cdot \underline{V}_{ph} = 0)$$

so Liouville Eqn. for $f(t, x, y)$ is:

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} = 0$$

$$\therefore \quad \frac{\partial f}{\partial t} - \frac{\partial H}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial f}{\partial y} = 0$$

Can further specialize: $H = H_0(x) + \tilde{H}(x,y)$

$$\Rightarrow \quad \begin{aligned} \dot{x} &= -\partial \tilde{H}/\partial y \\ \dot{y} &= \frac{\partial H_0}{\partial x} + \frac{\partial \tilde{H}}{\partial x} \end{aligned}$$

and

$$-\frac{\partial f}{\partial t} + \frac{\partial H_0}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \tilde{H}}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial x}\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + V_y(x)\frac{\partial f}{\partial y} + \{\tilde{H}, f\} = 0$$

e.g. $\begin{cases} H = \phi(r,\theta) \\ \text{G.C. plasma} \end{cases}$

b) Equation for Magnetic Flux $\psi(r,\theta)$

$$\underline{B} = B_0\hat{z} + \underline{\nabla}\psi \times \hat{z} \rightarrow \underline{B} \text{ field}$$

$\psi = A_z(r,\theta)$

$$\psi = \langle\psi(r)\rangle + \tilde{\psi}(r,\theta) \rightarrow \text{Magnetic flux function}$$

then, by definition:

$$\underline{B}\cdot\underline{\nabla}\psi = 0$$

(flux constant along magnetic field lines)

$\Rightarrow$

$$\left(B_0\frac{\partial}{\partial z} + \frac{B_\theta(r)}{r}\frac{\partial}{\partial\theta} + \underline{\tilde{B}}_\perp\cdot\nabla_\perp\right)\psi = 0$$

$\therefore$

$$\left(\frac{\partial}{\partial z} + \frac{1}{Rq(r)}\frac{\partial}{\partial\theta} + \frac{\tilde{B}_\perp\cdot\nabla_\perp}{B_0}\right)\psi = 0$$

$\to$, can read off analogy: (isomorphism)

$$\begin{cases} z \leftrightarrow t, \quad r \leftrightarrow x, \quad r\,d\theta \leftrightarrow y \end{cases}$$

$$\begin{cases} 1/R\,\mathfrak{q}(r) \leftrightarrow V_y(x) \leftrightarrow \omega(I) \\ \dfrac{\partial \omega(I)}{\partial I} \neq 0 \Rightarrow \mathfrak{q}'(r) \neq 0 \quad \text{"shear"} \end{cases}$$

(winding rate varies with radius)

$$\begin{cases} \langle V_y(x) \rangle \leftrightarrow B_\theta(r) \end{cases}$$

$$\begin{cases} \tilde{B}_\perp \leftrightarrow \nabla \tilde{H} \times \hat{z} \end{cases}$$

$$\begin{cases} \underline{\nabla}_\perp \cdot \underline{\tilde{B}}_\perp = 0 \\ \nabla \cdot (\nabla \tilde{H} \times \hat{z}) = 0 \end{cases}$$

Liouville Thm.
($\underline{\nabla} \cdot \underline{B} = 0$ underlies Hamiltonian structure)

Thus, Hamiltonian trajectory on 2-torus in phase space (for 2 degs. freedom) equivalent to trajectory of magnetic field line on torus of radius (minor) $= r$ in space!

$$\begin{cases} r \leftrightarrow I \\ 1/R\,\mathfrak{q}(r) \leftrightarrow \omega(I) \\ \theta \leftrightarrow \theta \quad \text{(angle variable)} \end{cases}$$

An Observation

Consider solution of flux equation perturbatively

i.e.

$$\underline{B} = \underline{B}_0 + \underline{\tilde{B}} \qquad \underline{B}_0 = B_0 \hat{z} + B_0 \hat{\theta}$$

$$\psi = \langle \psi(r) \rangle + \tilde{\psi}$$

$\therefore$

$$\underline{B} \cdot \nabla \psi = 0 \Rightarrow$$

$$(\underline{B}_0 \cdot \underline{\nabla}) \tilde{\psi} = - \tilde{B}_r \frac{\partial}{\partial r} \langle \psi(r) \rangle$$

expand ψ, $\tilde{B}_r$ as:

$$\tilde{B}_r = \sum_{m,n} \tilde{B}_r(r)_{\substack{m \\ n}} e^{i(m\theta - n\phi)}$$

$$z \to R\phi$$

$\Rightarrow$

$$\left(-\frac{in}{R} B_0 + \frac{im}{r} B_\theta \right) \tilde{\psi}_{\substack{m \\ n}} = - \tilde{B}_{r\substack{m \\ n}} \frac{\partial \langle \psi(r) \rangle}{\partial r}$$

$$\tilde{\psi}_{m,n}(r) = \frac{- \tilde{B}_{r\substack{m \\ n}}(r)\, \partial \langle \psi(r) \rangle / \partial r}{-i \frac{B_0}{R} \left(n - \frac{m}{q(r)} \right)}$$

$$\therefore \quad \tilde{\Psi}_{m,n}(r) = \frac{i R \left(\tilde{B}_{r\,m,n}(r) / B_0 \right) \partial \langle \Psi(r) \rangle / \partial r}{\left(n - \frac{m}{q(r)} \right)}$$

$\Rightarrow$ perturbative solution diverges at
$q(r) = m/n$ $\Rightarrow$ defines resonant surface (special tori)

i.e. radius where pitch of field line ($q(r)$) resonates with pitch of perturbation (m/n)

$\Rightarrow$ linear solution to Liouville Eqn. fails here.

$$\underline{n} \cdot \underline{\omega} = 0$$

(resonances)

Consequences of Phase Space Structure of Integrable Systems: Discussion and Examples

- consider integrable system with n degs. of freedom

then:

- can identify dimensions (generic)
    1) phase space : $d_p = 2n$
    2) energy shell : $d_E = 2n-1$
    3) torus : $d_T = n$

and

n	1	2	3	etc
d_p	2	4	6	
d_E	1	3	5	
d_T	1	2	3	

$\therefore$

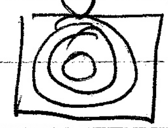
$\Rightarrow$ for 1D, energy shell and tori have $d=1$

$\Rightarrow$ tori fill energy shell

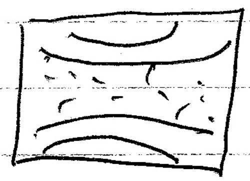
$\therefore$ 1D systems ergodic (fill allowed pieces of Γ)

n=2 ; 2D tori embedded in 3D energy shell

∴ integrable system ⇒ $\begin{cases}\text{tori fill energy shell} \\ \text{nested}\end{cases}$



non-integrable system ⇒ $\begin{cases}\text{gaps between} \\ \text{nested tori}\end{cases}$



⇔ $\begin{cases}\text{trajectory in gap can't} \\ \text{escape}\end{cases}$

→ n=3, can escape ⇔ Arnold diffusion.

- Motion on Tori

→ can generalize tokamak field-line representation to write

$$q_i(t) = \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} a^{(i)}_{k_1, \cdots k_n} e^{i(k_1\theta_1 + \cdots k_n\theta_n)}$$

$$= \sum_{k_1} \cdots \sum_{k_n} a^{(i)}_{k_1, \cdots k_n} e^{i(k_1\omega_1 + k_2\omega_2 + \cdots k_n\omega_n)t} e^{i\phi}$$

$k_1, \cdots k_n$ → quantum #'s for periodic motion (i.e. $k_1 = 1, 2 \ldots$)

$$\lambda^i_{\underline{k}}(\underline{J}) = \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \, \underset{\substack{\uparrow \\ \text{in } A^2 \text{ variables}}}{q_i} \, e^{i(k_1\theta_1 + \cdots k_n\theta_n)}$$

thus, $q_i(t)$ on torus is:

- multiply periodic
- $\sum_\alpha k_\alpha \omega_\alpha \neq 2\pi$ $\Rightarrow$ quasi-periodic orbit
  (not on resonant surface) $\Rightarrow$ orbit fills surface of torus <u>ergodically</u>

- i.e. for $n=2$; $\omega_1/\omega_2 = m/n$ rational
  {closed orbit
  {(small denominator)
  $\Rightarrow$ line does <u>not</u> fill 2-torus, rather closes on self.

but $\omega_1/\omega_2 =$ irrational
$\Rightarrow$ open orbit, so line ergodic on 2-torus

<u>Observe</u>: Since rationals are set with $\mu \equiv 0$ on real #, tori with open trajectories (ergodically covering tori) vastly outnumber resonant tori with closed orbits.

Secular Perturbation Theory

- strategy is to remove resonances via transformation to frame co-rotating with resonant variables

- akin removing resonance by frame change as in 1D particle motion

- limitation is removal one fast variable only possible,

Observe:

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

then if $r\omega_1 - s\omega_2 = 0$ (resonance)

$\Rightarrow$ $\theta = r\theta_1 - s\theta_2$ is "slow" variable

i.e. $\left(\underline{\omega} \cdot \dfrac{\partial}{\partial \underline{\theta}}\right) f(\theta) = \left(\omega_1 \dfrac{\partial}{\partial \theta_1} + \omega_2 \dfrac{\partial}{\partial \theta_2}\right) f(r\theta_1 - s\theta_2)$

$$= (r\omega_1 - s\omega_2)\frac{\partial f}{\partial \theta} = 0$$

near reson

$\Rightarrow$ f dependence on θ is higher order

i.e. slow.

Thus, in simplest case, canonical P.T. $\Rightarrow$

C.T.: $\underline{I}, \underline{\theta} \to \underline{J}, \phi$

here, canonical transform from 2 variables (both 'fast') to 1 slow, 1 fast variable

i.e.

$$\left.\begin{array}{l} I_1, \theta_1 \\ I_2, \theta_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} r\theta_1 - s\theta_2, & \hat{J}_1 \\ \theta_2, & \hat{J}_2 \end{array}\right.$$

(hats $\Rightarrow$ 2 variables)

i.e. same generic form as before, but eliminates 1 'fast' motion.

i.e. $F = S$ (old positions, new momenta)
$= S(\theta_1, \theta_2; \hat{J}_1, \hat{J}_2)$

$\Rightarrow$ type 2, with:

$$S = (r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2 \hat{J}_2 + \epsilon S_1$$

so

$$I_1 = \partial S/\partial\theta_1 = r\hat{J}_1 + \epsilon\,\partial S_1/\partial\theta_1$$

$$I_2 = \partial S/\partial\theta_2 = (\hat{J}_2 - s\hat{J}_1) + \epsilon\,\partial S_1/\partial\theta_2$$

$$\phi_1 = \partial S/\partial\hat{J}_1 = r\theta_1 - s\theta_2 + O(\epsilon)$$

$$\phi_2 = \partial S/\partial\hat{J}_2 = \theta_2 + O(\epsilon)$$

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

$$H_1 = \sum_{l,m} H_{l,m}(\underline{I})\, e^{i(l\theta_1 + m\theta_2)}$$

take $l, m \neq 0$

but know, $\phi_1 = r\theta_1 - s\theta_2$

$\phi_2 = \theta_2$

$$\therefore \begin{cases} \theta_1 = (\phi_1 + s\phi_2)/r \\ \theta_2 = r\phi_2/r \end{cases}$$

$\Rightarrow$ re-writing H_1:

$$H_1 = \sum_{l,m} H_{l,m}(\underline{\hat{J}}) \exp\left[i\frac{l}{r}(\phi_1 + s\phi_2) + im\phi_2 \right]$$

$$= \sum_{l,m} H_{l,m}(\underline{\hat{J}}) \exp\left[i\left(\frac{l}{r}\phi_1 + \frac{(ls + mr)}{r}\phi_2 \right)\right]$$

Now, $\phi_2 \rightarrow$ fast dependence

$\phi_1 \rightarrow$ slow dependence

$\therefore$ average out ϕ_2 dependence. CRITICAL to note averaging only valid near resonance (distinguishes fast, slow dependence)

enabled near resonance, only.

Thus, here:

$$K_1 = K_1(\hat{\underline{J}}, \overset{\text{slow}}{\phi_1}) = \langle H_1 \rangle_{\phi_2}$$ i.e. avg. out fast dependence

$$= \left\langle \sum_{\ell,m} H_{1\,\ell,m}(\hat{\underline{J}}) \exp\left[i\frac{\ell}{r}\phi_1 + \frac{i}{r}(\ell s + mr)\phi_2 \right] \right\rangle_{\phi_2}$$

$\therefore$ $\ell s = -mr$ selected by avgd. sum!

$$\Rightarrow \frac{\ell}{m} = -\frac{r}{s} = \frac{\omega_2}{\omega_1} \Rightarrow \text{resonance}$$

$\therefore$

$$\langle H_1 \rangle_{\phi_2} = \sum_{p=0}^{\infty} H_1^{(-pr,\,+ps)} e^{-ip\phi_1}$$

$$= \sum_{p=0}^{\infty} H_{1\,\substack{-rp\\+sp}} e^{-ip\phi_1}$$; i.e. sum over all harmonics of resonant pair

so averaged Hamiltonian:

$$\langle H \rangle = H_0(\hat{\underline{J}}) + \epsilon \sum_{p=0}^{\infty} H^{(1)}_{\substack{-pr\\ps}} e^{-ip\phi_1}$$

Now $\frac{\partial \langle H \rangle}{\partial \phi_2} = 0 \Rightarrow \frac{d}{dt}\hat{J}_2 = 0$

but from C-T. rules:

$I_1 = rJ_1$

$I_2 = \hat{J}_2 - s\hat{J}_1$

$$\boxed{I_i = \frac{\partial S}{\partial \Theta_i}}$$

$\Rightarrow \hat{J}_2 = I_2 + \frac{s}{r} I_1$

$\therefore$ have identified modified (adiabatic) invariant $\underbrace{\hat{J}_2 = I_2 + \frac{s}{r} I_1}_{\text{invariant of avgd. Hamiltonian}}$ via transformation

$\Rightarrow \frac{d\hat{J}_2}{dt} = 0 \Rightarrow \frac{d\hat{\Theta}_2}{dt} = \frac{\partial \langle H \rangle}{\partial \hat{J}_2} = \omega_{\text{avg.}}(\hat{J}_2)$

Now $\langle H \rangle = \langle H(\hat{J}_1, \phi_1; \hat{J}_2, \cancel{\hat{\phi}_2}) \rangle$

For full solution, need understand $\hat{J}_1, \hat{\phi}_1$ motion.

Now, can, without loss of generality, simplify remaining calculation by:

- assuming $p = 0, \pm 1$ harmonics <u>only</u> contribute to ϕ_1, $\hat{J}_1$ evolution

i.e. $$\langle H \rangle = H_0(\vec{J}) + \epsilon H_{0,0}(\vec{J}) + 2\epsilon H_{n,-s}(\vec{J}) \cos\phi_1$$

$$= H_0(\vec{J}) + \epsilon H_{0,0}(\vec{J}) + 2\epsilon H_{n,-s}(\vec{J}) \cos\phi_1$$

($H_{-n,s} = H_{n,-s}$; many pblms. feature 1 relevant harmonic, only).

- seek fixed points, frequency motion about fixed pts. (i.e. $\langle H \rangle$ is effectively that of oscillator/pendulum) $H = \frac{1}{2} I \dot{\theta}^2 + mgl(1-\cos\theta)$
$= \frac{1}{2} P_\theta^2/ml^2$

[General procedure: Fixed pts. and proximal (linear) stability]

$\therefore$ fixed pts:

$\partial \langle H \rangle / \partial \phi_1 = 0 \Rightarrow \dot{\hat{J}}_1 = 0$ ($\dot{\hat{J}}_2 = 0$ already)

$\partial \langle H \rangle / \partial \hat{J}_1 = 0 \Rightarrow \dot{\phi}_1 = 0$

these define $\left.\begin{array}{l}\hat{J}_{1,0} \\ \phi_{1,0}\end{array}\right\}$ $\left\{\begin{array}{l}\text{fixed pts.} \\ \text{of motion}\end{array}\right.$

-t,

$$\frac{\partial \langle H \rangle}{\partial J_1} = 0 \Rightarrow \frac{\partial H_0(\vec{J})}{\partial \vec{J}_1} + \epsilon \frac{\partial H_{0,0}(\vec{J})}{\partial \vec{J}_1}$$

$$+ 2\epsilon \frac{\partial H^{(1)}_{r,-s}}{\partial \vec{J}_1} \cos \phi_1 = 0$$

$$\frac{\partial \langle H \rangle}{\partial \phi_1} = 0 \Rightarrow -2\epsilon H^{(1)}_{r,-s} \sin \phi_1 = 0$$

$$\phi_1 = 0, \pm\pi, \ldots$$

Now,
$$\frac{\partial}{\partial \vec{J}_1} = \frac{dI_1}{d\vec{J}_1}\frac{\partial}{\partial I_1} + \frac{dI_2}{d\vec{J}_1}\frac{\partial}{\partial I_2}$$

$$= r\frac{\partial}{\partial I_1} - s\frac{\partial}{\partial I_2}$$

$\Rightarrow$ have fixed pt. conditions:

$$0 = \left(r\frac{\partial}{\partial I_1} - s\frac{\partial}{\partial I_2}\right) H_0(\mathbf{I})$$

$$+ \epsilon \frac{\partial}{\partial \vec{J}_1} H_{0,0} + 2\epsilon \frac{\partial H^{(1)}_{r,-s}}{\partial \vec{J}} \cos\phi_1$$

$$= (r\omega_1 - s\omega_2) + \epsilon\left(\frac{\partial H^{(1)}_{0,0}}{\partial \vec{J}_1} + \frac{2\partial H^{(1)}_{r,-s}}{\partial \vec{J}_1}\cos\phi_1\right)$$

(resonance! ~ 0)

note, to lowest order, $\Rightarrow$

$0 = r\omega_1 - s\omega_2$ $\Rightarrow$ $\hat{J}_{1,0}$ defined by resonant surface condition

exe: - field lines on torus $\Rightarrow$ $q(r) = m/n$

$r = q^{-1}(m/n)$

resonant, rational torus radius

- wave-particle interaction

$\frac{\omega}{k} = V$

and $-2\epsilon H^{(1)}_{r,-s} \sin\phi_1 = 0.$

Now, as seek simplified Hamiltonian in form of pendulum, convenient to expand H about $\hat{J}_{1,0}$, i.e.:

$$\langle H(\hat{J}_1, \phi_1)\rangle = \left[H_0(\hat{J}_{1,0}) + \epsilon H^{(1)}_{0,0}(\hat{J}_{1,0})\right] + \cancel{\left.\frac{\partial H_0}{\partial \hat{J}_1}\right|_{\hat{J}_{1,0}} (\hat{J}_1 - \hat{J}_{1,0})} + \frac{1}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 \left.\frac{\partial^2 H_0}{\partial \hat{J}_1^2}\right|_{J_{1,0}}$$

$(\text{defn. } \hat{J}_{1,0})$ $+ 2\epsilon H^{(1)}_{r,-s} \cos\phi_1$

$$\langle H(\vec{J}_1,\phi_1)\rangle = \frac{1}{2}(\vec{J}_1-\vec{J}_{1,0})^2 \underbrace{\left.\frac{\partial^2 H_0}{\partial \vec{J}_1^{\,2}}\right|_{\vec{J}_{1,0}}}_{\text{shear}} - F\cos\phi_1$$

$$= \frac{G}{2}(\vec{J}_1-\vec{J}_{1,0})^2 - F\cos\phi_1$$

$$G = \left.\frac{\partial^2 H_0}{\partial \vec{J}_1^{\,2}}\right|_{\vec{J}_{1,0}}, \quad F = -2\epsilon H^{(1)}_{l,-s}$$

Recall Pendulum : $\qquad L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

$$\therefore \quad H = p_\theta\dot{\theta} - L \qquad\qquad p_\theta = ml^2\dot{\theta}$$

$$= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta + \cancel{c}$$

$$\therefore \begin{cases} \langle H(\vec{J}_1,\phi_1)\rangle = \frac{G}{2}(\vec{J}_1-\vec{J}_{1,0})^2 - F\cos\phi_1 \\ H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta \end{cases}$$

$$\Rightarrow \boxed{\langle H(\vec{J}_1,\phi_1)\rangle = \frac{G}{2}(\vec{J}_1-\vec{J}_{1,0})^2 - F\cos\phi_1}$$

$\Rightarrow$ form of Hamiltonian near resonance

Note : Assumes $\frac{\partial^2 H_0}{\partial \vec{J}_1^{\,2}} = \frac{\partial \omega}{\partial \vec{J}_1} \neq 0$ $\left.\begin{cases} \text{NL} \\ \text{Shear}\end{cases}\right.$

corresponds to accidental resonance. $\partial^2 H_0/\partial \hat{J}_1^2 = 0$ (linear problem!) corresponds to intrinsic resonance.

$\Rightarrow$ Properties / Structure of $\left\{\begin{matrix}\text{Resonant}\\ \text{Standard}\end{matrix}\right\}$ Hamiltonian

$$\langle H(\hat{J}_1, \phi_1)\rangle = \underbrace{\frac{G}{2}}_{\text{NL parameter}}(\hat{J}_1 - \hat{J}_{1,0})^2 - \underbrace{F}_{\text{perturbation amplitude}}\cos\phi_1$$

$$= \frac{G}{2}\Delta\hat{J}_1^2 - F\cos\phi_1$$

and so

$$\begin{cases} \Delta\dot{J} = -F\sin\phi_1 \\ \dot{\phi}_1 = G\,\Delta J \end{cases}$$

$\phi_1 = 0 + \delta\phi_1$

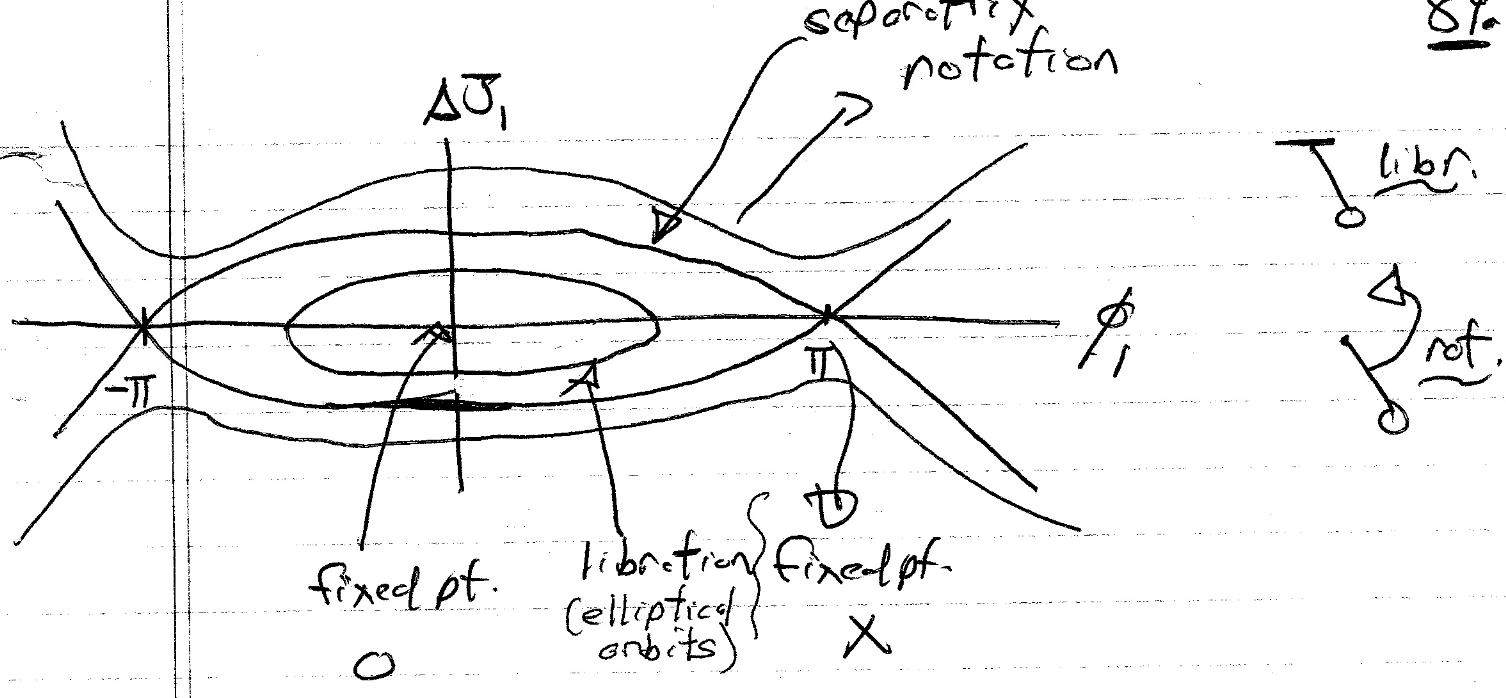
$\Rightarrow \Delta\ddot{J} + FG\,\Delta J = 0$

$\Rightarrow FG > 0 \Rightarrow \phi_1 = 0$ (O pt. / elliptic pt) fixed point stable

$\therefore$ $\phi_1 = \pm\pi$ (X-pt. / hyperbolic pt.) fixed point unstable

$\Rightarrow$ can draw contours of constant H

i.e.



– separatrix 'separates' rotation from libration (open from closed contours)

– width of separatrix – "island width"

$$(\Delta J)_{max} \cong 2(F/G)^{1/2}$$
$$= 2\left(-2G\, H^{(1)}_{n,-s} \Big/ \frac{\partial^2 H_0}{\partial J_1^2}\Big|_{J_{1,0}}\right)^{1/2}$$

i.e. particle + wave

$$H = \overbrace{(p+m\omega/k)^2}^{(\Delta p)^2}/2m + q\phi_0 \cos kx$$

$$\Delta p \approx (q\phi_0 m)^{1/2}$$

$$\Rightarrow \Delta v \approx (q\phi_0/m)^{1/2} \rightarrow \text{trapping width}$$

Physics 200B P. Diamond

Summary of Secular Perturbation Theory

→ Purpose:

a.) Ordinary canonical perturbation theory seeks to 'approximately integrate' perturbed Hamiltonian by constructing canonical transformation $I, \theta \to J, \phi$ such that $\dot{J} = 0$, to some order in ϵ

input: $H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$

output: $K = H_0(\underline{J}) + \epsilon \langle H_1(\underline{J}) \rangle_\theta + \dots$

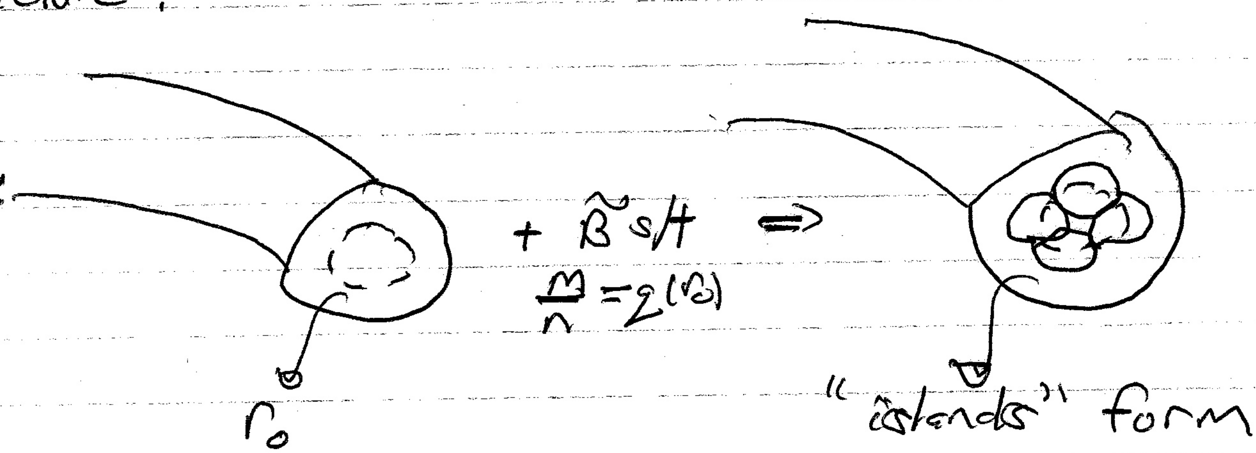
$$S = \underline{J} \cdot \underline{\theta} + \epsilon S_1, \quad S_1 = i \sum_{\underline{m}} \frac{\tilde{H}_{1,\underline{m}} \, e^{i\underline{m}\cdot\underline{\theta}}}{\underline{m} \cdot \underline{\omega}_0(\underline{J})}$$

b.) Secular (canonical) perturbation theory seeks to solve the "small divisor" problem, arising when $\underline{m} \cdot \underline{\omega}_0(\underline{J}) \cong 0$, i.e. on resonance. Apart from its specific tailoring to resonance, secular perturbation theory is very close to canonical perturbation theory in approach and technique.

N.B.: The "Big Picture":

Resonant perturbations distort phase space tori, breaking symmetry and forming structure.

i.e. tokamak:



seek calculate width, shape, etc. of tori distortions (i.e. islands) using secular perturbation theory.

⇒ Secular perturbation theory is important prelude / foundation for the study of Hamiltonian chaos.

→ Set Up:

① have: $H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$

with a resonant surface, i.e. $\exists$ some $\underline{I}_0$ such that (for 2D Γ)

$$r\,\omega_1(\underline{I}_0) - s\,\omega_2(\underline{I}_0) = 0$$

② then specialize usual $I, \theta \to \mathcal{J}, \phi$ by isolating slow, fast dependence:

$$\left.\begin{array}{l} I_1, \ \theta_1 \\ I_2, \ \theta_2 \end{array}\right\} \to \left\{\begin{array}{l} r\theta_1 - s\theta_2, \ \hat{\mathcal{J}}_1 \\ \theta_2, \ \hat{\mathcal{J}}_2 \end{array}\right. \quad \begin{array}{l} \text{(slow)} \\ \text{fast} \end{array}$$

so $$S = (r\theta_1 - s\theta_2)\hat{\mathcal{J}}_1 + \theta_2 \hat{\mathcal{J}}_2 + \epsilon S_1$$

Note: S.P.T. only works for 'one resonance at a time'.

③ Now, since: $$\begin{array}{ll} \phi_1 = r\theta_1 - s\theta_2 + \cancel{O(\epsilon)} & \text{(slow)} \\ \phi_2 = \theta_2 + \cancel{O(\epsilon)} & \text{(fast)} \end{array}$$

can re-write:

$$H_1 = \sum_{l,m} H_{l,m}(\underline{\hat{\mathcal{J}}}) \exp\left[i\left(\frac{l}{r}\phi_1 + \frac{(ls+mr)}{r}\phi_2\right)\right]$$

Now, in standard CPT eliminate all angle variable dependence by averaging. In SPT, resonant perturbations break angular symmetry, so we cannot remove all ϕ dependence. However, averaging allows us to remove fast angular dependence. ∴ average over ϕ_2.

$\Rightarrow \quad K_1 = K_1(\hat{\underline{J}}, \phi_1) = \langle H_1 \rangle_{\phi_2}$

$\therefore \quad H_1' \quad ls = -mr \quad \Rightarrow \quad \underbrace{\frac{l}{m} = \frac{-r}{s} = \frac{\omega_2}{\omega_1}}$

perturbation pitch must mach frequency ratio

$\Rightarrow$

$$K_1(\hat{\underline{J}}, \phi_1) = H_0(\hat{\underline{J}}) + \epsilon \sum_{p=0}^{\infty} H^{(1)}_{-pr,\, ps}\, e^{-ip\phi_1}$$

(sum over harmonics)

$\therefore \quad \frac{\partial K_1}{\partial \phi_2} = 0 \quad \Rightarrow \quad \frac{d\hat{J}_2}{dt} = 0$

$\Rightarrow \quad \hat{J}_2 = \underline{I}_2 + \frac{s}{r} I_1 \quad$ is I.O.M.

etc.

→ Understanding perturbed motion.

- no loss generality for $p = 0, \pm 1 \Rightarrow$

$$\langle H \rangle = H_0(\hat{\underline{J}}) + \epsilon H_{0,0}(\hat{\underline{J}}) + 2\epsilon H_{r,-s}(\hat{\underline{J}}) \cos \phi_1$$

- for understanding motion need, a la' harmonic oscillator, fixed point locations and effective "spring constants".

- Unsporting approach:
  - $\Rightarrow$ only $\hat{J}_1, \phi_1$ dependence non-trivial
  - $\Rightarrow \dot{\hat{J}}_1 = 0 \Rightarrow \phi = n\pi$ (pendulum equilibria)
  - $\dot{\phi}_1 = 0 \Rightarrow r\omega_1 - s\omega_2 = 0$ (resonance)

- Sporting:

$$\partial/\partial\hat{J}_1 = \frac{dI_1}{d\hat{J}_1}\frac{\partial}{\partial I_1} + \frac{dI_2}{d\hat{J}_2}\frac{\partial}{\partial I_2}$$

$$= r\,\partial/\partial I_1 - s\,\partial/\partial I_2 \quad , \text{etc}$$

$\Rightarrow$ $J_{1,0}$, $\phi_{1,0}$ fixed points: on resonant torus at $\phi = n\pi$

Then, can simplify K_1 to that of pendulum by expanding about $J_{1,0}$, $\phi_{1,0}$:

$$K_1 = \boxed{H_0(\hat{J}_{1,0}) + \epsilon H_{0,0}^{(1)}(J_{1,0})}$$

$$+ \underbrace{\cancel{\frac{\partial H_0}{\partial \hat{J}_1}}}_{\text{resonance}} (\hat{J}_1 - \hat{J}_{1,0}) + \frac{1}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 \left.\frac{\partial^2 H_0}{\partial \hat{J}_1^2}\right|_{\hat{J}_{1,0}}$$

$$+ 2\epsilon H_{r,-s}^{(1)} \cos\phi_1$$

$$\cong \frac{1}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 \frac{\partial^2 H_0}{\partial \hat{J}_1^2} + 2\epsilon H_{r,-s}^{(1)} \cos\phi_1$$

→ Output:

If $G \equiv \partial^2 H_0 / \partial \hat{J}_1^{\,2}\Big|_{\hat{J}_{1,0}}$, $F \equiv -2 G\, H^{(1)}_{r,-s}(\hat{J}_{1,0})$

$$\Rightarrow \boxed{K_1 = \langle H(\hat{J}_1, \phi_1)\rangle = \frac{G}{2}(\hat{J}_1 - \hat{J}_{1,0})^2 - F\cos\phi_1}$$

- form of H near resonance $r\omega_1 - s\omega_2 = 0$
- a la' pendulum
- requires: $\partial^2 H_0 / \partial \hat{J}_1^{\,2}\Big|_{\hat{J}_{1,0}} \neq 0$ (shear)

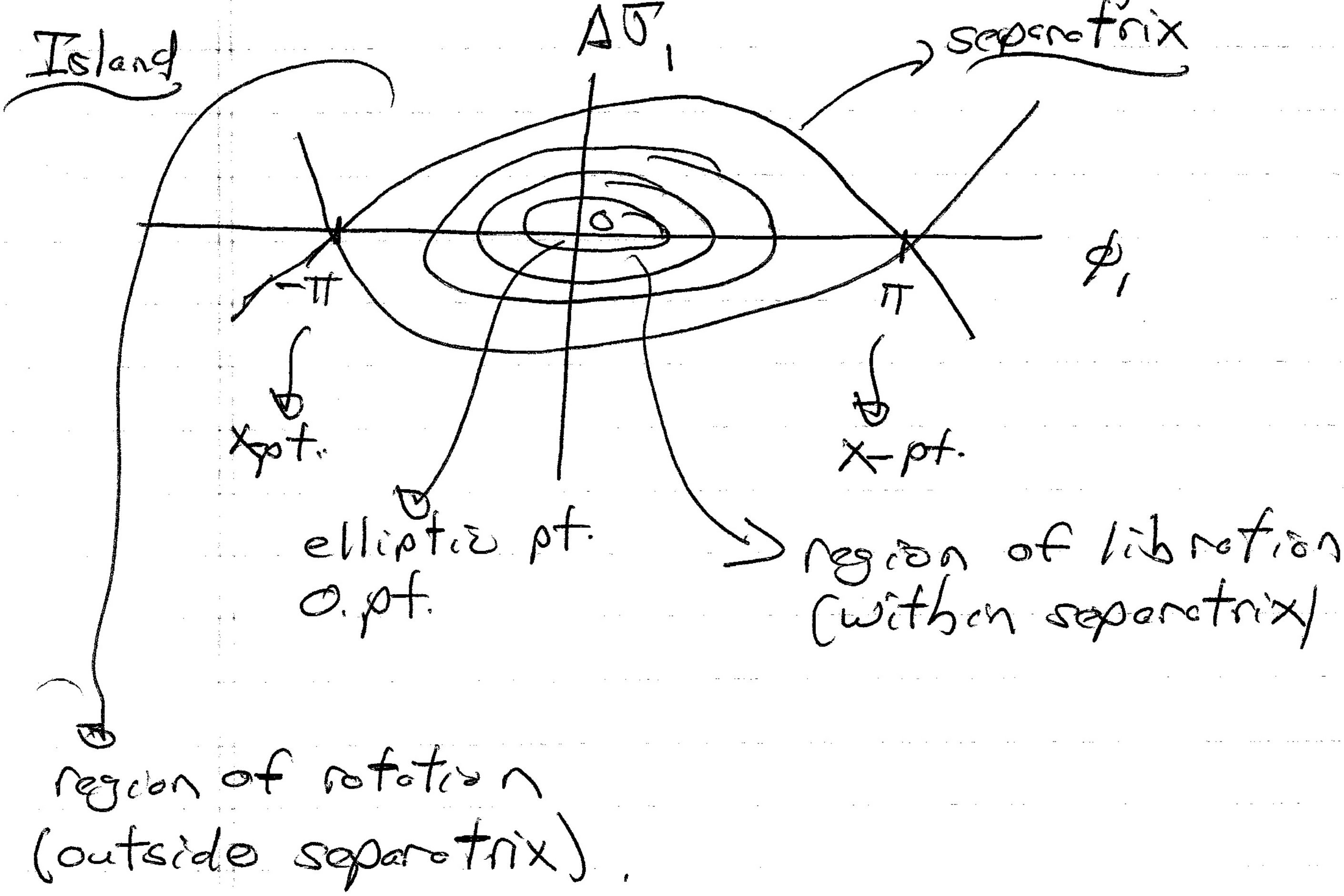
$$\langle H \rangle = \frac{G}{2}(\Delta \hat{J}_1)^2 - F\cos\phi_1$$

$FG > 0 \Rightarrow \phi_1 = 0$; stable fixed point
o-pt; elliptic

$\phi_1 = \pm\pi$ fixed point, unstable
X-pt; hyperbolic

$\Delta \hat{J}_{max} = 2\,(F/G)^{1/2}$ → 'island' width
$\sim (H_1)^{1/2}$, generically

→ The Picture (Const. H contours):



i.e. resonant perturbations form island chains on resonant tori