Physics 161: Black Holes: Lecture 7: 29 Jan 2013

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7.1 Shooting Light into a Black Hole

Let's calculate radial motion into black holes another way, a way that is often very useful, because it bypasses the geodesic equations. Let's calculate how long it takes to shoot a ray of light into black hole.

For light we know the invarient interval $ds^2 = 0$, that is the metric distance, aka proper time, is 0. So for light itself how long does it take to get into a black hole? It takes the same time it take light to go anywhere, zero! From light's point of view no time ever passes since $d\tau = 0$ always. OK, that means we set the metric equal to zero for light. This is the very useful trick. Considering radial infall, we can also set $d\phi = d\theta = 0$, and get

$$dt = \frac{dr}{(1 - r_S/r)}.$$

Integrating both sides from $t = 0$ at $r = r_0$ to $t = t$ at $r = r$, we find

$$t = r_0 - r + r_S \ln\left(\frac{r_0 - r_S}{r - r_S}\right).$$

From our previous work we expect the time for light go from $r = 30$ km to $r = r_S$ to be less than a millisecond. But plugging into the above equation we get a factor $\ln((30-8.85)/(8.85-8.85)) \rightarrow \ln(\infty) \rightarrow \infty$. There is a logarithmic divergence and again we see we never get into the black hole!

Of course, from the point of view of the light ray, or the falling guy, they get in and are crushed within a millisecond. It is just that time viewed from far away runs differently at the horizon of a black hole. The problem is with the use of this far away time. Also, while it is true that everything that ever falls into a black hole seems to "hang up" at the horizon, it is not that case that someone looking closely at a black hole sees all that junk. Remember the light coming to you from the falling objects redshifts to infinity and therefore those objects become invisible very quickly.

As an interesting aside consider what would happen if you tried to lower yourself slowly into a black hole on a very strong rope. I'm not going to do the calculation but the effective acceleration of gravity you would feel would increase without limit. The effective g, which is $GM/r^2 = 9.8$ m/s^2 here on Earth becomes

$$g = (GM/r^2)(1 - r_S/r)^{-1/2}.$$

Thus the force becomes infinitely strong at $r = r_S$, the rope will break and you will fall to your death.

7.2 Inside the Black Hole

The Schwarzchild metric is the solution to Einstein's GR equations in the vacuum around a spherically symmetric object. Thus we expect these to work even if the object is smaller than r_S. What happens at and inside $r = r_S = 2GM/c^2$? The metric is bad at $r = r_S$, but actually it is OK inside.

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

First consider lightcones, the causally connected past and future. How did we find these in Special Relativity? Considering just one space and one time dimension: $ds^2 = -dt^2 + dx^2 = 0$, for light. This defines the null geodesics. Solving we find $dt = \pm dr$, or $dt/dr = \pm 1$, which implies the lightcones are lines of 45^0 in a spacetime diagram. Remember also that above the 45^0 lines is the timelike $ds^2 < 0$ future, while outside the lines is the $ds^2 > 0$ spacelike elsewhere. In Special Relativity all lightcones have these 45^0 lines.

How about in General Relativity. Again we set $ds^2 = 0$ to find the null geodesics, giving $-dt^2(1 - r_S/r) + dr^2/(1 - r_S/r) = 0$, or

$$\frac{dt}{dr} = \pm\frac{1}{1 - \frac{r_S}{r}}.$$

Thus the lightcones are not 45^0 lines. As $r \to \infty$, $r_S/r \to 0$, so $dt/dr \to \pm 1$, and the lightcones are at 45^0, but closer to the spherical mass the angles are smaller, $dt/dr > \pm 1$. Thus the lightcones squeeze-up. Thus the path of light as it travels towards a spherical mass in a spacetime diagram is not on a 45^0 line, but on a curved line that gets more steep as it approaches the Schwarzschild radius.

Fig: Path of light as it approaches a black hole

We can calculate the angle of the lightcone at any value of r very simply. Just use

$$\frac{dt}{dr} = \tan\theta.$$

Thus at 30 km from a $3M_\odot$ mass hole the angle of the lightcone is $dt/dr = 1/(1 - 8.85/30) = 1.42$, or $\theta = 55^0$. This angle goes to 90^0 at $r = r_S$. So we see that right at $r = r_S$, $dt/dr \to \infty$, and no progress in r is possible! Not even light can make it into the black when far-away time is used as the coordinate. How does this jive with the fact that we calculated things can fall into black holes in just milliseconds? The point is that t is time measured by someone far away, not the traveler.

Now look what happens to the metric when $r < r_S$. Notice that since $r_s/r > 1$, the terms $(1 - r_S/r)$ become less than zero.

$$ds^2 = -dt^2(1 - r_S/r) + dr^2/(1 - r_S/r) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Thus we see that the t coordinate and the r coordinate terms switch signs! Remember that in GR r is not a distance, it is a coordinate. You have to use the metric to find distances. Same with t. The minus sign is how we recognize the time coordinate in GR, so this means that r becomes timelike and t becomes spacelike! If you try to find the lightcones inside the black hole you see they flip over sideways!

Fig: Lightcones around and inside a black hole

The same force that moves eveyone towards the future now moves things towards $r = 0$, which is the future lightcone! This one way of understanding the reason that once inside the black hole you can't get out. There is no force in nature that can move you backward in time. Thus inside a hole there is no force in nature that can move something towards larger r! Note that while proper time increases, t actually decreases. This is not a problem since t is no longer a time dimension. It is a space dimension; r is the time. We can see this explicitly by considering two points inside, say near $r = 2$ km and $r = 2.1$ km, so $dr = 0.1$ km. Take $dt = 0$ and calculate the invariant interval: $ds^2 = +dr^2/(1 - 8.85/2) = -0.0029$ km < 0, that is the interval is timelike, and thus these points can be causally connected! Similarly, two events with $dr = 0$ and $dt \neq 0$ have $ds^2 > 0$ and therefore are spacelike separated and cannot be causally connected. Moving straight up in the spacetimem diagram is not allowed.

The angle of the lightcone just inside r_S is still 90^0, but rather than being squeezed, it is now wide open. As you move in towards $r = 0$, the lightcone closes up, but continues to point toward $r = 0$. Inside $dr/dt = \pm(1 - r_S/r)$. As $r \to 0$, $dr/dt \to \pm\infty$, or $dt/dr \to 0$, and the cone squeezes in pointing towards the center of the hole. So while at first when you are inside the hole there some freedom of movement, in the end you are directed directly at the center.

7.3 Orbits in the Schwarzschild metric

Let's go back to our geodesic equations and find the orbits around a black hole or other spherical object. So far we only considered radial orbits. To extend this to circular and eliptical orbits we will use the method of effective potentials, which allows one discover the main types of orbits and the main points of interest without doing the hard work of actually solving the differential equations. This is very general technique and worth learning.

7.4 Effective Potential for Newtonian Orbits

First let's do it in Newtonian mechanics, by writing the total energy $E = T + V = \frac{1}{2}mv^2 - GM/r$ in spherical coordinates. Recall that in 3-D non-relativistic mechanics, $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, where $\dot{x} = dx/dt$, etc., and $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$. Let's simplify by considering motion in the x-y plane so $z = 0$, $\theta = \pi/2$, and $\dot{\theta} = 0$. Then $\dot{z} = 0$, $\dot{x} = \dot{r}\cos\phi - r\sin\phi\dot{\phi}$, and $\dot{y} = \dot{r}\sin\phi + r\cos\phi\dot{\phi}$. Substituting this into the formula for kinetic energy and using $\sin^2\phi + \cos^2\phi = 1$, we find:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

Next defining the angular momentum $l = mv_\phi r = m\dot{\phi}r^2$, so $r\dot{\phi} = l/(mr)$, we can write the total energy as

$$E = \frac{1}{2}m\left(\dot{r}^2 + \frac{l^2}{m^2r^2}\right) - \frac{GMm}{r},$$

or

$$E = \frac{1}{2}m(\frac{dr}{dt})^2 + V_{eff},$$

with

$$V_{eff} = -\frac{GMm}{r} + \frac{l^2}{2r^2m}.$$

Thus we have written the equation of motion as a one dimensional equation in the radial coordinate, r, with energy $E = T + V_{eff}$. The value of doing this is that we are very familar with the the solutions of a one dimensional equation for a particle moving in a potential well with the shape V_{eff}. Depending on the total energy, the particle may escape the well and travel off to infinity, or it might be trapped in the well and oscillate back and forth in the bottom of the well. For small enough energy it might be at rest at the bottom. So we now think of the radial coordinate r as the one dimension and can easily understand the possible Newtonian orbits around a spherical object.

Fig: Effective potential for Newtonian Potential

To discover the possible orbits we draw a plot of r vs. the effective potential. We see that V_{eff} drops from infinity at $r = 0$, reaching a minimum at a value of $r = r_{min}$, and then rises slowly to $V_{eff} = 0$ at $r = \infty$. The exact shape depends upon the constants (conserved quantities) l, and E, as well as M.

The easiest solution to this one dimensional problem is the particle as rest at the minimum. The particle just sits there, which means $r =$ constant. This is the circular orbit of Kepler's laws. The radius of the circular orbit is found by finding the minimum of the effective potential by solving $dV_{eff}/dr = 0$. This gives $0 = GMm/r^2 - l^2/(mr^3)$, or $r = GM/v^2$, where we substituted back in $l = mvr$ before solving for r. Note that we did not substitute in for l before taking the derivative with respect to r, since l is a conserved quantity and therefore constant along the orbit. This is the result you get from $F = ma$, aka $GMm/r^2 = mv^2/r$. In this case, the total energy is at its minimum which is less than zero. $E_{min} = -mv^2 + \frac{1}{2}mv^2 = -\frac{1}{2}mv^2$. Note if you had a different value of v, that would imply a different value of l, and a different r_{circ}.

Another obvious solution is the particle having $E < 0$, so the particle oscillates around the bottom between turning points r_1 and r_2 where $E = \frac{1}{2}m\dot{r}^2 + V_{eff}$. These orbits are the elliptical Kepler orbits, and the position and speed of the orbiting object at turning points can be found from the equation above. There are also orbits that have $E > 0$ which can come in from infinity, reach a point of closest approach and then return to infinity. These are the hyperbolic Kepler orbits. Finally, a particle with $E = 0$ will do a similar similar thing, but the orbit will be parabolic.

7.5 Effective Potential for Schwarzschild Orbits

Let's apply the same effective potential technique to the full r geodesic equation we derived earlier.

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right),$$

where $r_S = 2GM/c^2$. Here the effective potential is

$$V_{eff} = \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right) = m - \frac{r_S}{r}m + \frac{l^2}{mr^2} - \frac{r_S l^2}{mr^3},$$

and keep in mind that we are using the proper time τ, not t as the time variable, and l is the angular momentum and E is the total energy. We can spot the different types of orbits as we did for the Newtonian effective potential. Now we see there are five different types of orbits.

Fig: Effective potential for Schwarzschild Geodesics

1. At the minimum we have a bound circular orbit. The radius is found from $dV_{eff}/dr = 0$ as before.

2. We also have the bound orbits with turning points r_1 and r_2 like in the Newtonian case, but here these turn out to not be elliptical! The orbits don't close. If we do the perhelion advance of Mercury we will show this. The time to between r_1 and r_2 and back is more than the time to go around $\phi = 360^0$, so the perhelion advances.

3. Next, as for Newtonian hyperbolic orbits, we have orbits that start at $r = \infty$, come in, reach a distance of closest approach, and go out again. Close examination shows these are close to, but not exactly the same as, the Newtonian case.

4. Now look at maximum point on top of the hill. We find this point also when we set $dV_{eff}/dr = 0$, since the slope is zero here. If you set a particle there and carefully balanced it, it would stay there. Thus this is another circular orbit, but it is unstable. A tiny perturbation outward and the particle will escape to infinity. A tiny perturbation inward and it will fall in the hole.

5. Finally, there is another new type of orbit not found in the Newtonian case, the capture orbit. If the energy is high enough, the particle goes over the high point and then plunges down into the hole, never to return. In the Newtonian case a particle can never be captured in this way due to gravity alone, even though this is how we think when, for example, a comet hits the Sun. Hitting the Sun's surface invokes non-gravitational forces; a point mass object could never capture anything in Newtonian mechanics.

Let's find the radii of the circular orbits. We can write

$$V_{eff} = m + \frac{l^2}{mr^2} - \frac{r_S m}{r} - \frac{r_S l^2}{mr^3},$$

and differentiate and set to zero.

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{mr^3} + \frac{r_S m}{r^2} + \frac{3r_S l^2}{mr^4} = 0.$$

We can simplify this to a quadratic equation in r by multiplying through by mr^4:

$$r_S m^2 c^2 r^2 - 2l^2 r + 3r_s l^2 = 0,$$

which has two solutions:

$$r_\pm = \frac{l^2}{r_S m^2 c^2}\left(1 \pm \sqrt{1 - \frac{3r_S^2 m^2 c^2}{l^2}}\right),$$

where I've put the c's back in for fun and reference.

Note that if the quantity in the square root is negative we don't have a real solution. Thus there is no circular orbit in that case. This happens when the angular momentum is too small, i.e. there are only capture orbits. If the quantity in the square root in positive, then we have two solutions, that is, two circular orbits at different radii, as we saw in the plot. Thus we have two circular orbit solutions if $l^2 > 3r_S^2 m^2 c^2$, and none if not.

When there are two orbits we can tell if the orbits are stable or unstable by taking the 2nd derivative. If $d^2V_{eff}/dr^2 > 0$, the the curvature of the effective potential is positive and it is a minimum. This means the orbit is stable. On the other hand if if $d^2V_{eff}/dr^2 < 0$, it means there is a maximum in the effective potential at that point, and the orbit is unstable. We can do the math, but we saw before from the picture that the smaller solution (the one with the minus square root) was unstable, and the larger solution with the plus square is a stable orbit.

From the above discussion we see that the smallest possible *stable* circular orbit will happen when the square root term vanishes. This happens when $1 = 3r_S m^2 c^2/l^2 = 0$, or $l^2 = 3r_S^2 m^2 c^2$. The value of the radius for this value of the angular momentum can be found just by plugging this value of l into the formula for $r_\pm$, and we find that the minimum stable circular radius is

$$r_{min} = \frac{l^2}{r_S m^2 c^2} = 3r_S,$$

that is just exactly three times the Schwarzschild radius. Actually we should check that this orbit is stable, since for this value of l^2, the stable and unstable orbits are the same. Plugging in r_{min}, and the value of l^2 above into

$$\frac{d^2 V_{eff}}{dr^2} = \frac{6l^2}{mr^4} - \frac{2r_S m}{r^3} - \frac{12 r_S l^2}{mr^5},$$

we find that $d^2 V_{eff}/dr = 0$ at this combined minimum. Thus this orbit is just really neither stable nor unstable, but neutral. It is called, however, the minimum or last stable orbit, since an orbit just a tiny bit larger is in fact stable.

This last stable orbit value is a very important result. When things fall into black holes they have trouble getting in because of angular momentum. The tend to get ripped apart and form what are called accretion disks. The material in the disk gradually spirals inward. However, when the radius reaches $3r_S$, there is no longer a stable circular orbit, so the material all just flows into the hole. Thus we expect that when we look at real black holes in space we will see material down to $3r_S$ but not any closer. When we detect X-rays from black holes, we hope we are looking at material radiating from the distance $3r_S$. And when we find periodic signals from black hole candidates, we assume this is the radius at which the material is orbiting.