Physics 161: Black Holes: Lecture 6: 24 Jan 2013

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6 Distances and Times around a Black Hole

Suppose you have a very powerful spaceship and fly near to a black hole. Can you notice anything different from deep space? (Actually the calculations and results we get here are also true around the Earth on a very small scale.)

Suppose we fly close to a small black hole of mass $M = 3M_\odot$. (Actually, the smallest black hole we expect to find in nature is around $3M_\odot$, so we pick this number.) We know the Schwarzschild radius for this black hole is about 3×2.95 km $= 8.85$km, so we are careful to stay farther away from the hole than this. Let's suppose we fly all the way around the hole and measure a distance around (circumference) of $C = 2\pi 30$km $= 188.5$ km. How far are we from the hole? Naively, we expect we are 30 km from the hole, but we should check this by using the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

To find the proper distance in the θ direction, we set $dt = dr = d\phi = 0$, and the find the proper length

$$dl_\theta = \sqrt{ds^2} = r d\theta.$$

Note that this is the same as the proper distance in flat space! So there is no curvature in the θ (or ϕ) direction! To find the distance around the black hole we integrate

$$C = \int_0^{2\pi} dl_\theta = 2\int_0^{\pi} r d\theta = 2\pi r,$$

just as in flat space. Thus if we have measured the distance around as 2π 30 km, we know we are at radial coordinate $r = 30$ km.

But does that mean we are 30 km from the center of the hole? We have to use the metric again to find out. This time we want the radial direction, so we set $dt = d\theta = d\phi = 0$, and we get

$$dl_r = \sqrt{ds^2} = \frac{dr}{\left(1 - \frac{2GM}{r}\right)^{1/2}}.$$

To find the distance from radial coordinate r_1 to radial r_2 we integrate from r_1 to r_2. If we set $r_1 = 8.85$ km and $r_2 = 30$ km, we can find our distance to the black hole horizon itself.

$$\Delta l_r = \int_{r_1}^{r_2} dr \left(1 - \frac{2GM}{r}\right)^{-1/2}.$$

Defining

$$A_i \equiv \sqrt{1 - \frac{2GM}{r_i c^2}} = \sqrt{1 - r_S/r_i},$$

we evaluate the integral as

$$\Delta l_r = \int_{r_1}^{r_2} dl_r = r_2 A_2 - r_1 A_1 + \frac{r_S}{2} \ln\left(\frac{r_2 A_2 + r_2 - r_S/2}{r_1 A_1 + r_1 - r_S/2}\right),$$

where we used the Schwarzschild radius $r_S = 2GM/c^2 = 2.95326 M/M_\odot$ km.

Using this formula we can find the distances. For example starting at r=30 km, and moving in to r = 20 km, we we naively expect to move 10 km, but we actually move 12.51 km. This is very weird, since after moving 12.51 km inward from $r = 30$km, the distance around the black hole would be $2\pi 20$ km *not* $2\pi 17.49$ km. Thus we are seeing directly the curving of space. It is just like the example of the two surveyors, but now the curvature is not in any direction we can experience! In the above, the way we tell what value of r we are at is to travel around the hole and use the circumference. Using the above formula we also find the distance from $r_1 = 10$ km to $r_2 = 30$ is 29.50 km, and the distance from the horizon at $r_S = 8.86$ km to $r = 30$ km is 35.98 km.

Fig: embedding diagram of curved space near a black hole

We can visualize the spatial curvature around a black hole by an embedding diagram. The key in this diagram is that the radial coordinate is just the straight 3-D distance, while the proper distance is measured along the curved surface. We continue the embedding diagram even inside the black hole horizon, which turns out to be correct, though we will have to think carefully before understanding why.

Finally, note that this metric and embedding diagram work not only for black holes, but also for the Earth! If the distance to the center of the Earth is $r_{\rm Earth}$, then the distance around the Earth is really not equal to $2\pi r_{\rm Earth}$! Can you figure the difference?

6.1 Can you fall into a black hole?

We found above that the space is flat in the θ and ϕ directions, but curved in the radial direction with proper distance $dl_r = dr/\sqrt{1 - r_S/r}$. Looking at this again we see that moving a small proper distance dl_r implies moving a smaller radial coordinate distance $dr = dl_r\sqrt{1 - r_S/r}$. This seems fine, but look at what happens right near the Schwarzschild radius r_S. When $r \to r_S$, then $dr \to 0$, that is you are not moving at all in the radial coordinate! Does this mean that you can't get into the black hole? No, because it is a square-root singularity, which implies that it is integrable.

Consider the area under the curve $y = 1/\sqrt{x}$. As $x \to 0$, $y \to \infty$, but still the area is $A = \int_0^{x_0} x^{-1/2} dx = 2x^{1/2}|_0^{x_0} = 2\sqrt{x_0}$ which is finite. Likewise $dl/dr = (1 - r_s/r)^{-1/2}$ is integrable and we gave the formula for the integration above. We therefore find that the proper distance from $r = 30$ km to r_S is 35.98 km. So just because a function goes to infinity in a region of interest doesn't always mean there is a problem. Of course, sometimes it does mean there is a problem as we shall see.

6.2 Time to fall into a black hole

Next let's calculate the time to fall into a black hole. There are several ways that we will do this. First, let's return to our geodesics for radial infall. Remember we considered the case with angular momentum

$l = 0$, and found that starting at rest from $r = \infty$ meant the conserved energy was $E = mc^2$, and the equation of motion is

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0.$$

Let's find the proper time τ it takes for this freely falling object to go from $r_0 = 30$km to $r = r_S = 8.85$ km, the Schwarzchild radius of a 3 $M_\odot$ black hole. This is the time as measured by the wristwatch of the falling guy. We take the negative square root of this equation since we want to fall in (r decreasing):

$$\frac{dr}{d\tau} = -\sqrt{2GM}r^{-1/2},$$

or

$$\int_{30}^{r} dr r^{1/2} = \int_0^{\tau} -\sqrt{2GM}d\tau,$$

or

$$\frac{2}{3}r^{3/2}|_{r_0}^{r} = -\tau\sqrt{2GM},$$

or

$$\tau = \frac{2}{3c}(1/\sqrt{2GM/c^2})(r_0^{3/2} - r^{3/2}).$$

(Note I put the c back in to get the units right and make calculation easier.) So to go from $r_0 = 30$ km to $r = r_S = 8.85$ km in our $3M_\odot$ black hole takes $\tau = \frac{2}{3}(1/\sqrt{(2.95)(3)\text{km}})((30\text{km})^{3/2} - (8.85\text{km})^{3/2})/3 \times 10^5\text{km/s} = 1.03 \times 10^{-4}$ s. Thus it takes about 0.1 millisecond! It is not clear yet, but this same equation works inside the black hole, so we can also find how long the falling guy has to live before hitting the singularity at the center. Just taking $r = 0$ in the above equation gives

$$\tau = \frac{2}{3}\frac{r_0^{3/2}}{\sqrt{2GM}} = 0.124\text{ms}.$$

Thus our guy gets only an extra 0.021 ms to live inside the black hole!

Now this is the time freely falling starting at infinity. We could also find the time to fall in if we started from rest at $r = 30$ km. For this we go back to the more general geodesic equation before plugging in $E = mc^2$,

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)mc^2 = 0.$$

Now we want the boundary conditions starting at $\tau = 0$: $dr/d\tau = 0$ at $r = r_0$. Plugging this in, we find the conserved energy is $E = \pm mc^2\sqrt{1 - 2GM/r_0c^2}$. Note this is the quantity that is conserved along the geodesic, and is NOT $E = mc^2$. For the case of falling in we take $E < 0$ then integrate the above equation as before. This can be done and we find a more complicated formula. The time to fall into $r = 0$ is a little simplier:

$$\tau = \frac{\pi}{2}r_0\sqrt{\frac{r_0}{r_S}},$$

for a time of 0.29 ms, about three times longer than when you start falling from far away.

Now what does this look like to someone watching from far away. They don't use the proper time τ, but use the coordinate for "far away" time t. We can convert the equations above to coordinate time t by using our time geodesic equation: $\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m}$, or $dt/d\tau = (E/m)/(1 - r_S/r)$. Then

$$dr/dt = (dr/d\tau)/(dt/d\tau) = -\frac{m}{E}(r_S/r)^{1/2}(1 - \frac{r_S}{r}).$$

This equation can be solved as before, but we will find some trouble in doing it as we get close to r_S. Consider that limit, $r \to r_S$. Then $1 - r_S/r \to 0$, and a tiny step in dr means an infinite step in dt. This is different than the case before with proper distance because this is not an integrable square root singularity. This is real infinity that cannot be integrated over. In fact, if you try to do the integral you will find you *never* get to $r = r_S$! Time slows down and motion ceases. Everything hangs up at the horizon.