Physics 161: Black Holes: Lecture 4: 17 Jan 2013

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4 Geodesics: Moving in "straight lines" through curved spacetime

We saw that gravity curves time and space. A very important result is how things move in curved space. The basic principle is that things move on paths that are as "straight" as possible. These can be defined as the paths which extremize the distance between two points, and are called **geodesics**. In 3-D Euclidian geometry a straight line is defined as the shortest distance between two points, and Newton's law say in the absense of outside forces particles move along such lines. It is similar in GR, but sometimes it is the "maximum" invariant interval which is relevant, not the minimum. That is why we say "extremizes" the distance instead of minimizes.

To get an idea of this, consider the 2-D analogy of an ant crawling on the surface of an apple. The ant is forced to follow the curved 2-D space of the apple skin, but suppose it walks as "straight" as possible, i.e. not veering to the left or right. As it walks from one side of the apple to the other coming near the stem, the ant will be "deflected" and come away from the stem at a different angle than it approached the stem. If the ant is close to the stem it could even circle the stem continually while still following the "geodesic". This should seem similar to the analogy of the two surveyors.

Now consider two points on opposite sides of the stem and ask which path is the shortest between them (the geodesic). Note that without the curved surface the shortest path would be quite different. However, the metric requires us to stay on the surface.

If we want to calculate the motion of objects in GR, we need to be able to find the geodesics. By finding them we will be able to derive Newton's laws from the Schwarzschild metric, as well as Einsteinian corrections to Newton's laws. We can find out the real answer obeyed by actual objects in the solar system. We will also be able to find out how objects move around black holes.

4.1 Geodesics and Calculus of Variations

GR says that the motion of a particle that experience no external forces is a geodesic of the spacetime metric. One can summarize GR in two statements: 1. Matter and Energy tell spacetime how to curve. 2. Curved spacetime tells matter and energy how to move. In solving for the geodesics we are finding how matter and energy (light) move.

In 3-D Euclidian space the definition of a geodesic, aka "straight line" is the shortest distance bewteen two points. Mathematically this can be found from calculus of variations on the metric distance. The differential distance $ds = \sqrt{dx^2 + dy^2 + dz^2}$ can be integrated between two end points to find

$$s = \int_a^b ds = \int_a^b \sqrt{1 + (dy/dx)^2 + (dz/dx)^2} dx,$$

where we factored out a dx so the integral is done over the x-axis. Note now the integrand contains the derivatives of the the functions $y(x)$ and $z(x)$ which define the path. How do find the path (functions $y(x)$ and $z(x)$) that minimize s? We need a general method. This method is called the "calculus of variations" and you probably learned it in calculus. However, I will remind you of how it is done. If you haven't seen it before, that's ok, since I'll show you all you need. This is beautiful and fun math that is used throughout advanced physics.

The basic idea is to minimize ds the same way you minimize any function is calculus: take its derivative, set it to zero, and solve. This solution will be the geodesic, that is the extremal path. This method works on curved surfaces since ds measures the actual distance following the curved surface. It works the same way in GR; you just add the time part of the metric.

4.2 Geodesics as equations of motion

A geodesic on pure spatial manifold (e.g. curved surface) is a line. For example, straight lines on a plane, or great circles on a sphere surface. In GR the geodesics include time and so are actually the equations of motion! You can understand this by remembering the distinction between time and space. One has choice in spatial motion, but one is forced by nature to move forward in the time direction: 1 sec per sec. You will hit the year 2014 no matter what you do. Now in flat space a possible geodesic is one in which you don't move in space at all: $dx = dy = dz = 0$. Then $ds = dt$ is an extremal path and you just sit there getting older. However, if spacetime is curved, then motion in dt can *require* motion in dx! Think of the surveyor analogy, where motion north (z) required motion in the x and y direction to stay on the surface of the Earth. Thus since motion in t direction is forced, motion in the x (or other spatial) direction will also be forced. Thus you have the geodesic requiring $dx/dt \neq 0$. dx/dt is a velocity, so you can't stay at rest and be on the geodesics. This is how gravity attracts things and why geodesics near a mass require bodies to fall towards the mass center. Let's do some math.

4.3 Euler-Lagrange Equations

To get used to calculus of variations, let's do a calculation some of you have already done: Lagrange's equations in classical mechanics. Then we will do it on metrics. Doing it several times is necessary and I highly recommend you go over all these calculations at home several times.

We want to extremize the integral $s = \int ds$, which can be written $s = \int (ds/dt)dt$, where we have factored out a t. We can factor out any variable we want so as to make the integrating easier. As an example, consider the general "least action" integral

$$S = \int L dt,$$

where L is called the Lagrangian. In our example $L = ds/dt$, but in regular classical mechanics the Lagrangian is $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$, where $\dot{x} = dx/dt = v_x$, T is the kinetic energy, and $V(x)$ is the potential energy. (Here we used only the x and t dimensions.) Thus for 1-D classical mechanics $S = \int_a^b (\frac{1}{2}m\dot{x}^2 - V(x))dt$ is called the action, and the equations of motion, $F = ma$, with $F = -dV/dx$

are found minimizing this action (principle of "least action"). In our problem we are trying to find the path that gives the shortest (or longest) distance along the path between two fixed points.

Let's do it first in general and get the equations.

$$S = \int L(x, y, z, \dot{x}, \dot{y}, \dot{z}, \lambda) d\lambda,$$

where we explicitly put in the velocities v_x, v_y, and v_z as variables that have to be solved for, and have factored out the general variable λ (usually just time). This lambda is called the "affine parameter" and is the variable you use to trace along the geodesic path. In the above, the dot means differentiating with respect to λ, i.e. $\dot{x} = dx/d\lambda$. Now we take the "variation" of this integral δS using the chain rule and set it to zero: $\delta S = 0$, and solve.

$$\delta S = \int \left(\frac{\partial L}{\partial x} \delta x(\lambda) + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial y} \delta y + \cdots \right) d\lambda = 0.$$

Next note that $\delta(\dot{x}) = \delta \frac{dx}{d\lambda} = \frac{d}{d\lambda} \delta x$, where $\delta x(\lambda)$ is a small deviation from the extremal path. Now integrate by parts the term with the $\delta \dot{x}$, by using:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du.$$

$$\int_a^b \frac{\partial L}{\partial \dot{x}} \delta \dot{x} d\lambda = \int_a^b \frac{\partial L}{\partial \dot{x}} \frac{d}{d\lambda} (\delta x) d\lambda = \frac{\partial L}{d\dot{x}} \delta x|_a^b - \int_a^b \frac{d}{d\lambda} \frac{\partial L}{d\dot{x}} \delta x d\lambda.$$

We want a path defined by $x(\lambda)$, $y(\lambda)$, etc. that goes from point a to point b. Thus we want $\delta x(\lambda)$, the variation of the path from the geodesic to be zero at the end points. That is $\delta x(\lambda = a) = 0$ and $\delta x(\lambda = b) = 0$ (and similarly for δy and δz). Thus the first term on the right hand side of above equation vanishes, and the equation becomes:

$$0 = \int \left(\frac{\partial L}{\partial x} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}} \right) \delta x d\lambda + \text{y and z parts}.$$

The final step in getting our Euler-Lagrange equations is to note that the variation in the path $\delta x(\lambda)$, $\delta y(\lambda)$, and $\delta z(\lambda)$ is completely arbitrary. Thus for the integral as a whole to vanish, the integrand itself must vanish everywhere. That is, to be true for every possible function $\delta x(\lambda)$, the parts of the intregrand multiplying δx, δy, and δz must be zero. Thus we have the **Euler-Lagrange equations**:

$$\frac{\partial L}{\partial x} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}} = 0.$$

$$\frac{\partial L}{\partial y} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{y}} = 0.$$

$$\frac{\partial L}{\partial z} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{z}} = 0.$$

4.4 First Example of Euler-Lagrange equations: classical mechanics

As a first example of the use of The Euler-Lagrange equations, let the Lagrangian be $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$ and let $\lambda = t$. Then $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F$, where we used the normal definition of force as the derivative of the potential energy. Since $v = \dot{x}$, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, and $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = ma$, where the acceleration $a = dv/dt$. Thus the Euler-Lagrange equations found by extremizing the action is just $F = ma$. This might seem like a lot of work to get something you already know, but the beauty of the method is that it works in difficult situations and in difficult coordinate systems. It is usually a lot easier to write down the kinetic and potential energy than it is to use the vector form of $F = ma$ in complicated situations.

4.5 Second example of Euler-Lagrange equations: Flat space geodesics

Now let's extremize the 3-D flat space metric to see if the shortest distance between two points is indeed a straight line! So $s = \int ds$, with

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + \dot{y}^2 + \dot{z}^2}dx,$$

where we have chosen $\lambda = x$, and $L = \sqrt{1 + \dot{y}^2 + \dot{x}^2}$. The y Lagrangian equation thus reads:

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0,$$

or

$$\frac{d}{dx}\left(\frac{1}{2}(1 + \dot{y}^2 + \dot{z}^2)^{-1/2}2\dot{y}\right) - 0 = 0,$$

or

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = \text{constant.}$$

For the z equation we find similarly,

$$\frac{\dot{z}}{\sqrt{1 + \dot{y}^2 + \dot{z}^2}} = \text{constant.}$$

Dividing the y equation by the z equation we get $\dot{y}/\dot{z} =$ constant, so $\dot{z} = c_1\dot{y}$. Substituting this into the y equation, we get $\dot{y}/\sqrt{1 + \dot{y}^2 + c_1^2\dot{y}^2} =$ constant. Since the only variable in this entire equation is $\dot{y}$, solving this equation for $\dot{y}$ will give a constant. Thus we find the Euler-Lagrange equations for the extremal distance between two points are $dy/dx = m_y$, and $dz/dx = m_z$, where m_y and m_z are some constants found by the boundary condition. Thus $y = m_y x + b_y$ and $z = m_z x + b_z$, the equation for a straight line in 3-D. This again might seem like a lot of work to prove that the shortest distance between two points is a straight line, but the method is general.

Before going on to extremize the invariant interval in a spacetime metric, I want to do the last problem again and show you a useful trick. In the above we used x as the affine parameter. Instead we could have used s itself. Thus we write our integral as $s = \int ds$, with the Lagrangian $L = 1$! However we write this 'one' in a special way:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dx}{ds})^2}ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}ds,$$

where now $L = 1 = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. As before in the Euler-Lagrange equations the term $\partial L/\partial x = 0$, and similarly for the y and z equations, so they reduce to:

$$\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) = 0,$$

and similarly for the y and z equations. But since $L = 1$ is a constant it comes out of the differential, becoming

$$\frac{d}{ds}\left(\frac{\dot{x}}{L}\right) = d\dot{x}/ds = 0.$$

This says that $\dot{x} = dx/ds =$ constant, or $x = m_x s + b_x$, and similarly, $y = m_y s + b_y$, and $z = m_z s + b_z$. Again the equation for a straight line, with s being the distance traveled along the line, but with less algebra.

4.6 Geodesics in Minkowski spacetime

Next we use the Euler-Lagrange equations to extremize the invariant interval, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, between two events in Minkowski spacetime. This will give us the equations of motion of Special Relativity! Let's use proper time τ as the affine parameter: $d\tau = \sqrt{-ds^2}$. We write $s = \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\, d\tau$, where $\dot{t} = dt/d\tau$, $\dot{x} = dx/d\tau$, etc. Since basically $\tau = s$, here again the Lagrangian $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$.

Noting that $\partial L/\partial t = 0$, and similarly for $\partial L/\partial x = 0$, etc. the Euler-Lagrange equations become

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} = 0,$$

and

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} = 0,$$

and similarly for y and z. Thus as we move along the path (varying τ to move along the geodesic) we have 4 conserved quantities, $\frac{\partial L}{\partial \dot{t}} =$ constant, $\frac{\partial L}{\partial \dot{x}} =$ constant, etc. Explicitly, these give $\partial L/\partial \dot{t} = \dot{t}/L = \dot{t} =$ constant, and $\partial L/\partial \dot{x} = \dot{x} =$ constant, so our equations are $\dot{t} = c_t$, $\dot{x} = c_x$, $\dot{y} = c_y$, and $\dot{z} = c_z$, or $t = c_t\tau + t_0$, $x = c_x\tau + x_0$, etc., where the c_t,c_x, t_0, x_0, etc. are constants.

To find the values of the constant let's simplify to only the x direction and let time t start with $\tau = 0$, so $t = c_t\tau$. Then $x = c_x\tau + x_0 = (c_x/c_t)t + x_0$, and we recognize the combination $c_x/c_t = v_x$ the velocity in the x direction. Thus $c_x = c_t v_x$. Now evaluate the Lagrangian $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2} = \sqrt{c_t^2 - c_x^2} = \sqrt{c_t^2 - c_t^2 v_x^2}$, or $1 = c_t\sqrt{1 - v_x^2}$. Thus $c_t = 1/\sqrt{1 - v_x^2} = \gamma$, the Lorentz factor, and we have as our geodesics the equations of special relativity: motion in a straight line with time dilation included: $t = \gamma\tau$, $x = v_x t + x_0$, etc.

Note that it is possible to have all the constants $v_x = v_y = v_z = 0$, so just $t = \tau$ (standing still aging) is a geodesic.

In summary, what did here is extremize (in fact maximize) the proper time between two events to find the geodesics. Thus the geodesic is that path for which the **maximum** time passes on the wrist watch of the observer traveling that path. [It is maximum, rather than minium due to the minus sign in the metric.] Note that this is basically the answer to twin paradox. The twin that went out and then back did not travel a geodesic, they accelerated three times, while the stay at home twin did not accelerate and therefore followed a geodesic. Thus we understand why the stay at home twin ages more (in fact ages maximally!). Whenever someone accelerates, they leave their geodesic and therefore are aging less! The solution to the twin paradox is also easily understood using a spacetime diagram. Both twins start together at the origin. One twin stays on Earth (worldline is geodesic going straight up). The other accelerates close to the speed of light (worldline close to 45^0). The proper time for the speedy twin is very small; remember the hyperbola of constant proper time. Halfway out, the speedy twin accelerates again (leaves the geodesic again), and speeds home, meeting up with the first twin. From this it is clear that if one minimized the proper time, it would require accelerating to the speed of light for half the time and then coming back with a total proper time approaching 0. We see that actual geodesics maximize the proper time.

4.7 Conserved quanties in the Euler-Lagrange formalism: Energy and Momentum

Note that as we did the derivations of the geodesics above we came across quantities that did not change as we traced along the affine parameter, that is we found conserved quantities. This is a general and important thing to watch for in using the Euler-Lagrange formalism. In general there will be a conserved quantity whenever the Lagrangian L does not depend explicitly on one of the variables. Actually there will be one conserved quantity per variable. This is easy to see. To be general, let's use the 4-vector notation x_μ, where $x_0 = t$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, so μ runs from 0 to 3 and x_μ can represent any of the spacetime variables. In this notation, all the Euler-Lagrange equations can be written in one line:

$$\frac{\partial L}{\partial x_\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial d\dot{x}_\mu} = 0.$$

So if L does not depend on one variable, call it x_μ, we have $\frac{\partial L}{\partial dx_\mu} = 0$, and the Euler-Lagrange equation reads: $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}_\mu} = 0$. Thus along the geodesic (running over values of the affine parameter λ), the quantity $\frac{\partial L}{\partial \dot{x}_\mu}$ is a constant.

In general we define p_μ, the **conjugate momentum**, of a variable x_μ, as

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}.$$

Thus we see that if the Lagrangian does not depend explicitly on a variable, then that variable's conjugate momentum is conserved. This p_μ may be an actual momentum, but it could be some other conserved quantity. If one takes $L = \frac{1}{2}m\dot{x}^2$, then in fact $p_x = mv_x$, which is conserved since L does not depend upon x. If instead we took $L = \frac{1}{2}m\dot{x}^2 - V(x)$, then the conjugate momentum is still $p_x = mv_x$, but it is not conserved along the geodesic as the affine parameter (t) changes. Of course this is because there is a force due to the potential energy.

For the Minkowski metric, let's see what these conjugate momenta are. We saw above that the metric did not depend explicitly on any of the x_μ; t,x, y, or z, thus we expect to have 4 conserved quantities. Consider the t equation with affine parameter proper time τ. We have $p_t = \frac{\partial L}{\partial \dot{t}} = \dot{t}$. Likewise for the x equation we find $p_x = \dot{x}$, $p_y = \dot{y}$, etc. During the above calculation we found that the constant for the Euler-Lagrange equations (which we called c_t) was equal to the Lorentz factor γ. Thus we see that

$$p_0 = p_t = \gamma = E/m,$$

where we noticed that in special relativity the energy of a particle is $E = m\gamma$, where m is the rest mass. Thus the conserved quantity associated with the t variable is nothing other than the energy. (It is energy per unit mass actually, but since p_t is a constant we can multiply by another constant m and still have a constant.)

We also previously found the momentum conjugate to the x variable, $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} = c_x$. Using $c_x/c_t = v$ from before, we have $p_x = v_x c_t = v_x p_t = v_x \gamma$. But in special relativity the momentum is just $mv\gamma$, so see that $p_x = P_x/m$, is the momentum per unit mass. (We call the actual momentum P to distinguish it from the momentum per unit mass.) Thus the 4 conserved quantities we discovered are just the energy and 3 components of momentum.

We can find the relation between these quantities from our definition of the Lagrangian: $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$. Substituting in $p_t = \dot{t} = E/m$, $p_x = \dot{x} = P_x/m$, etc. we find

$$1 = \sqrt{(E/m)^2 - (P_x/m)^2 - (P_y/m)^2 - (P_y/m)^2}.$$

Squaring, multiplying through by m^2, and using $P^2 = P_x^2 + P_y^2 + P_z^2$, we find the well known result $E^2 = P^2 + m^2$. Finally using units to put back the c's, $E^2 = P^2c^2 + mc^4$, which reduces to the famous $E = mc^2$ in the limit of zero velocity (zero momentum).