

Physics 161: Black Holes: Lecture 3: 15 Jan 2013

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3 General and Special Relativity

GR is based upon **the equivalence principle: All local inertial frames are equivalent for performance of all physical experiments.** Remember that a **local inertial reference frame is one that is free falling and non-rotating**, where particles move at constant velocities in straight lines (or stay still). A very important point is that Special Relativity is how spacetime behaves in any inertial frame. This is equally true for someone out in deep space far from anything, for someone orbiting the Earth in a spaceship, or for someone falling in an elevator with broken cables. All three someones are in inertial reference frames and would find the same result for any experiment they did; that is they could not tell which of the situations they were in, without looking outside (looking outside makes the frame bigger, i.e. non-local). The influence of the Earth's gravity on the guy falling in the elevator is not measurable by local experiments.

The metric of Special Relativity (SR) is the Minkowski metric $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$, and GR is an extension of SR. So we start by understanding the SR metric. In SR the relations between any two local inertial frames are the Lorentz transformations which you have seen before. These, and all of SR, can be derived from two postulates:

1. The principle of relativity: results of experiments don't depend on velocity of lab.
2. All inertial frames measure the same speed of light which is **exactly** $c = 299792458$ m/s! This postulate is the same as requiring the invariant interval ds to be the same in every frame.

These postulates can be used to derive the Lorentz transformations.

3.1 Spacetime diagrams

A very nice tool is the spacetime diagram. These will be indispensable when we try and go inside a black hole, so it is worthwhile trying to understand them.

Figure: Intro to Spacetime diagram

Points:

- Axes are x vs. t . Points are events, lines represent observers moving through spacetime (worldlines) or extended objects (if spacelike).
- The slope of an observer's worldline is $dt/dx = 1/v$.

- With $c = 1$, an observer moving at the speed of light is a 45° line. Nothing can move at a shallower slope, since that would mean $v > c$. Thus only events within the “lightcone” can be casually connected. [We suppress the y and z directions; if we include y then it is a cone.]
- Straight lines represent inertial observers (each related by a Lorentz transformation.) A line straight up is someone standing still in the lab frame.
- Curved lines represent accelerated observers.

3.2 More special relativity: how time and space appears to other observers: Lorentz transformation

Now let’s use spacetime diagrams to understand some of the weirdness of special relativity. Suppose observer O uses (t, x) and is in the lab, while observer O' uses (t', x') and is moving in the x -direction with velocity v .

Figure: O' moving at v in O coordinates

Where do the t' and x' coordinate axes go in this diagram? Well, the worldline of O' is their t' axis, since this is $x' = 0$ to them (and also $y' = z' = 0$). Where is the x' axis? To find this consider O' spacetime diagram from O' ’s point of view.

Figure: O' spacetime diagram in O' coordinates: reflecting light

From O' view t' axis goes straight up and x' axis is at right angles. Consider a photon emitted at $(t', x') = (-a, 0)$ in the $+x'$ direction. It reaches the x' axis at $t' = 0$, and then is reflected back (by a mirror). It returns to the t' axis ($x' = 0$) at $(t', x') = (a, 0)$. Consider other photons emitted at different times, $-b$, etc. and we see that we could *define* the x' axis as the locus of points that will reflect photons emitted at $t' = -a$, to reach back at $t' = a$.

This should be a frame-independent definition, so we can find the x' axis in the O frame using it.

Figure: t' and x' axes in O coordinates: reflecting light

To do this we use postulate two: light travels at $c = 1$ in all frames, that is, the light pulses O' uses to find the x' axis travel at 45° also in the O spacetime diagram. We see that the x' axis is *not* the same as the x axis! So events which are simultaneous to O' (i.e. happen at the same values of t' , e.g. $t' = 0$ on the x' axis), are not simultaneous to O (don’t happen at same values of t , i.e. are not parallel to the x axis.) This is the famous failure of simultaneity in SR, and is made very obvious in spacetime diagrams.

With some work, from the above postulates one can derive the famous **Lorentz transformations**:

$$\begin{aligned}
 t &= \gamma(t' + \frac{v}{c^2}x') \\
 x &= \gamma(vt' + x') \\
 y &= y' \\
 z &= z',
 \end{aligned}$$

where the Lorentz factor is the greek letter gamma:

$$\gamma = 1/\sqrt{1 - v^2/c^2}.$$

Note that with $c = 1$ these simplify:

$$t = \gamma(t' + vx')$$

$$x = \gamma(vt' + x')$$

$$y = y'; z = z',$$

with $\gamma = 1/\sqrt{1-v^2}$.

One can easily find the inverse transforms, just by setting $v \rightarrow -v$ in the above: **Inverse Lorentz transformation:**

$$t' = \gamma(t - vx)$$

$$x' = \gamma(-vt + x)$$

In the homework you will use these to prove that Δs between two events is the same in every frame.

3.3 Calibrating the axes in a spacetime diagram: hyperbolas=circles!

Figure: Hyperbolas on space time diagram

How does one calibrate the axes in a spacetime diagram? Consider the equation: $t^2 - x^2 = a^2$. All the points on this hyperbola have the same invariant “distance” from $(0, 0)$. $\Delta s^2 = -\Delta t^2 + \Delta x^2 = -a^2$; e.g. $x = 0$ and $t = a$, etc. Thus this hyperbola is the equivalent of a circle in normal Euclidian geometry. The proper time is the time that someone traveling along the line would measure on their wristwatch. So no matter what the speed, all travelers from the origin to the hyperbola take the same time to get there (i.e. they age the same amount, a). Note that $-a^2$ is the invariant interval, and since $ds^2 = -a^2 < 0$ this hyperbola is for timelike separations.

Similarly one can plot the equation: $t^2 - x^2 = -b^2$, at get the sideways hyperbola which represents spacelike separations. These points all have the same proper length from the origin.

Figure: Events A and B on space time diagram

To calibrate the axes, we consider two events, A and B. A lies at $(t, x) = (1, 0)$ on the $O t$ axis, while event B lies at $(t', x') = (t', 0)$ on the $O' t'$ axis. We pick B to lie along the hyperbola so the invariant interval (proper time) from the origin is the same to both events A and B. Since the interval Δs is invariant (same in all frames), and event A has $t = 1, x = 0$, we have $\Delta s^2 = -t^2 + x^2 = -1$. Therefore in moving frame, $-t'^2 + x'^2 = -1$ also. Event B occurs at $x' = 0$, which implies $t' = 1$. But on the spacetime diagram, the distance from the $(0,0)$ to event B looks farther than the distance from $(0,0)$ to event A! The point here is that one can't use Euclidian intuition about distances on spacetime diagrams: It is Minkowski geometry. The proper distance is $\Delta s^2 = -\Delta t^2 + \Delta x^2$, *not* $\Delta t^2 + \Delta x^2$ as we are used to. Now you see the importance of the minus signs in the spacetime metric. They change the geometry from Euclidian to Minkowski.

In the above, the moving frame observer, O' , says $t' = 1$, but the lab frame says the clock tick happened at a longer time. (Just read off the t time from the t axis!)

This says that the moving person's time runs slowly, when observed from the lab frame.

Finally, note that on a space time diagram we can easily see which events are separated by time-like intervals and which are separated by space-like intervals. Thus we can see which events can and cannot be causally connected. **The 45° lines are called the lightcone.** If we add say the y dimension you see why it is called a cone, but if we add the y and z dimensions it is actually a cone in 4-D! The cone opening towards larger time is called the future lightcone and contains all future events that can be influenced by it. The cone opening towards the past is call the past lightcone and contains all the events that could have influenced the event at the apex of the cones. You can draw a lightcone around any event and see which events can influence which other events.

3.4 Time dilation and length contraction by spacetime diagram

Last time we saw how to calibrate spacetime diagrams. We can use that result to derive the famous time dilation: moving clocks run slow to stationary observers.

Numerically we can calculate for time dilation for event B, considering starting a clock at the origin (event A). $\Delta s^2 = -\Delta t'^2 + \Delta x'^2 = -\Delta t^2 + \Delta x^2$. For a clock tick at event B, $\Delta t = t$, $\Delta x = x$, $\Delta t' = t'$. In the O , lab frame: Event B occurs $x = \Delta x = v\Delta t$, and time occurs at $t = \Delta t$. In O' , the moving frame, $\Delta x' = x' = 0$ since events B and A occur at the same place (O' is not moving in its own frame). So $-\Delta t'^2 + 0 = -\Delta t^2 + \Delta t^2 v^2 = -(1 - v^2)\Delta t^2$, or $\Delta t = \Delta t' / \sqrt{1 - v^2} = \gamma \Delta t'$. This is the time dilation you learned before in special relativity.

One can similarly find out the length contraction you learned about in SR. If you remember the derivation, you will remember that it is more tricky than the time dilation calculation. The reason for the trickiness is easy to see in the spacetime diagram. Consider the origin as point A and a ruler of length a laid out on a spacecraft with its far end at point B. How does that look? It lies along the x' axis, which we have just seen how to find (the set of points which all have $t' = 0$). Now draw the sideways hyperbola of all proper distances a . If we just project the point B onto the x axis we find the wrong answer, that the ruler looks *longer* in the lab frame. What have we done wrong? By length of ruler in the lab frame we mean the distance between the two ends measured at the same lab frame time t . Thus the distance between points A and B is not the distance we are after. We need to follow point A up in time until it reaches the same time t as point B has. We know the t' axis has the same angle from the 45° line as the A-B segment, so we can just go along the t' axis until we get as high as point B. Now the length of the ruler as measured in the lab frame is the difference in x of the two ends measured at the same time. We see the ruler is shorter as measured in the lab frame as expected.

3.5 Other special relativity you need to know

This class has special relativity as a prerequisite, so I'm not going to cover much more. However, you should look over the book to refamiliarize yourself with several topics. For example, you should understand $E = mc^2$. In fact, you need to know that the real equation is $E^2 = (mc^2)^2 + (pc)^2$, where p is the momentum. With $c = 1$ this is more suggestively written: $m^2 = E^2 - p^2$, and we see that the rest mass is an invariant in much the same way that the interval ds is. You also should be familiar with relativistic energy and momentum: $E = m\gamma c^2$, and $p = m\gamma v$. Try Taylor expanding the relativistic expression for kinetic energy $T = E - mc^2 \approx \frac{1}{2}mv^2$, and find the next order correction to the famous Newtonian formula.

3.6 Time Dilation in a gravitational field

Remember the metric of flat spacetime is

$$ds^2 = -c^2 dt^2 + dx^2,$$

where we have suppressed the y and z dimensions. In this metric, we find the proper time by setting $dx = 0$ and using $d\tau = \sqrt{-ds^2}/c = dt$, and see that coordinate t is actually the proper time between events. Likewise the proper distance is $dl = dx$, so that the coordinate x is the proper distance. Compare this to the Schwarzschild metric

$$ds^2 = -c^2 \left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

Here the proper time is found by setting $dr = d\theta = d\phi = 0$. Note then $ds^2 \leq 0$, that is a time-like separation. The proper time is again

$$d\tau = \sqrt{-ds^2}/c = \sqrt{1 - \frac{2GM}{rc^2}} dt.$$

So if $dt = 1$ sec, $d\tau = \sqrt{1 - \frac{2GM}{rc^2}}$ sec, which is a smaller (i.e. shorter) time. This proper time is the clock time measured at a distance r from the Earth (or a black hole). Thus we see that clocks run slower in a gravitational field than in deep space (where $r \rightarrow \infty$ and $dt = d\tau$).

Let's see how big effect this is for us living on the surface of the Earth. $M_{\text{Earth}} = 6 \times 10^{27}$ gm, and $r_{\text{Earth}} = 6.38 \times 10^8$ cm, and a useful number to calculate is the Schwarzschild radius of the Earth $r_{SC} = 2GM_{\text{Earth}}/c^2 = 0.886$ cm. Then $2GM_{\text{Earth}}/r_{\text{Earth}}c^2 = .886/6.38 \times 10^8 = 1.4 \times 10^{-9}$. Using a Taylor expansion, this says $d\tau = \sqrt{1 - 1.4 \times 10^{-9}} \approx 1 - \frac{1}{2}1.4 \times 10^{-9}$. Or $d\tau = (1 - 6.9 \times 10^{-10})dt$. So time runs slower here on Earth (as measured from deep space by about 1 sec every 45 years! Not a big effect, but measurable. Gravity curves time.

However in other environments this factor can be bigger. Especially wierd is what it does when $r = r_{SC}$, Then $\sqrt{1 - 2GM/rc^2} \rightarrow 0$, and the measured time seems to stop! This occurs at the horizon of a black hole.

3.7 Old Idea of Black Holes

The idea of black holes has been around since the 1700's when Laplace, Mitchell, and others thought about giant stars. One can even derive the Schwarzschild radius using Newton's laws and get the right answer (by coincidence I think?) Consider a big spherical ball of mass M and radius r . The gravitational potential energy at the surface of the ball is $V = -GMm/r$, where m is just some small test mass. Since V goes to zero at infinity, the **escape velocity** from the surface is found by setting this potential energy to the kinetic energy $E = \frac{1}{2}mv^2$, and solving for v . This gives $v_{esc} = \sqrt{2GM/r}$. This escape velocity increases as the mass increases, or the radius decreases. And one could ask what happens when the escape velocity is equal to or larger than c . Then according to Newton's corpuscular theory of light (which is incorrect!) light would get trapped in such an object, and the giant star would be dark. Setting $v_{esc} = c$ one finds the radius of such an object is $r = 2GM/c^2$, precisely the Schwarzschild radius. This Newtonian calculation gives the right answer, but is wrong. For example, light particles are massless so $E \neq \frac{1}{2}mv^2$. There are several errors in this calculation which just happen to cancel. The correct description requires the Schwarzschild metric.