Physics 161: Black Holes: Lecture 12: 14 Feb 2013

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12 Where do metrics come from?

We pulled our Schwarzschild metric out of a hat. But to really find a metric you have to solve Einstein's general relativistic field equations. For a simple spherical situation in vacuum like the Schwarzschild case, this is not too hard, but in general it is nearly impossible. Einstein's equations are a set of 10 coupled partial differential equations that are made even more difficult because there are some symmetries, called general coordinate invariance that have to be obeyed. There are really only a very few analytic solutions known for realistic situations.

I want to write down these equations for you so you can see what they are and why it is difficult to solve them. We are not going to try to solve them. In order to understand these equations we will have to learn a little tensor notation. Tensor notation is a way to write complicated equations in a short hand. This is similar to writing Maxwell's equations using the div and curl. Maxwell himself did not have that convienent notation and the equations he wrote were much longer and more complicated to look at.

In tensor notation Einstein's field equations are very simple:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda \eta_{\mu\nu}.$$

We need to go over the terms in this equation slowly and step by step. First remember the idea of 4-vectors, like

$$x^\mu = (t, x, y, z),$$

or

$$p^\mu = (E, p_x, p_y, p_z).$$

Here the index μ is equal to 0,1,2, or 3, with $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $p^0 = E$, $p^1 = p_x$, etc. The zeroth component is the timelike component, and the first through third are spacelike.

Note that we can have indices both above and below the letter. The indices above the letter must not be mistaken for taking a power. When index is above the letter it indicates a 4-vector, when it is below the letter it indicates a 1-form.

In differential geometry one would learn exactly what differental forms are, and how they differ from 4-vectors, but here we will skip all that. One can create a 1-form corresponding to a 4-vector using a metric. I will show that in a minute, but in flat space, the 1-forms corresponding to the 4-vectors above are:

$$x_\mu = (-t, x, y, z),$$

and

$$p_\mu = (-E, p_x, p_y, p_z).$$

The only difference in this case is that the index is lower rather than upper, and the timelike component got a minus sign. If one used the other signature metric, then the time-like component got a minus sign.

The way you create these 1-forms is using the flat space Minkowski metric of special relativity:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We wrote this earlier as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, which technically is called the **line element**. The metric is actually a rank 2 tensor, that is the 4 by 4 matrix given above. We form the line element (what we have been calling the metric) by muliplying the 4 by 4 matrix by the two copies of four vector dx^μ:

$$ds^2 = \sum_{\mu\nu} dx^\mu \eta_{\mu\nu} dx^\nu.$$

Summing first over ν we get

$$\sum_\nu \eta_{\mu\nu} dx^\nu = (-dx^0, dx^1, dx^2, dx^3) = (-dt, dx, dy, dz) = dx_\mu.$$

Then the sum over μ is

$$ds^2 = \sum_\mu dx_\mu dx^\mu = -(dx^0)^2 + (dx^1)^2 + (dx^3)^2 + (dx^3)^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

that is, the line element above that we have been calling the Minkowski metric. Note that in order to get the 1-form (lower the index), we just multiply by $\eta_{\mu\nu}$ and sum over one of the indices. Thus in general in flat space we can use $\eta_{\mu\nu}$ to lower indices:

$$v_\mu = \sum_\nu \eta_{\mu\nu} v^\nu,$$

where v^μ is any 4-vector. We can also raise indices using $\eta^{\mu\nu}$ which is the same as $\eta_{\mu\nu}$. Raising and lowering works for all tensors. As another example try working out $\sum_\mu p_\mu p^\mu = -E^2 + p_x^2 + p_y^2 + p_z^2 = -m^2$. The beauty of the notation is that one can tell what an object is just from the unsummed over indices. If there are none, then the object is a "scalar", and is an invariant like the mass or proper distance; it won't change by a boost, translation, or rotation. If there is one upper index it is a 4-vector, and will tranform like x^μ (a simple Lorentz transformation). If there is one free lower index it means means a 1-form. Two lower indices means a 2-form, etc. and the transformation properties of these are well determined as well.

Now, most people use the **Einstein summation convention** which means to just leave out all the summation symbols! Any index that is repeated is summed over. For example then one writes the equation above as

$$v_\mu = \eta_{\mu\nu} v^\nu,$$

and then $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. We have been calling ds^2 the metric because it contains the same information as $\eta_{\mu\nu}$, the Minkowski, or flat space metric. We go from flat space to full curved space in GR by changing the elements of the 4 by 4 matrix. We use the symbol $g_{\mu\nu}$ for the full curved space metric and write the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

In spherical coorindates we don't use (t, x, y, z), but instead use (t, r, θ, ϕ) as coodinates for (x_0, x_1, x_2, x_3). Thus remembering the Schwarzschild metric we have $g_{00} = -(1 - r_S/r)$, $g_{11} = (1 - r_S/r)^{-1}$, $g_{22} = r^2$, and $g_3 = r^2 \sin^2 \theta$, with all $g_{\mu\nu} = 0$ when $\mu \neq \nu$. In curved space $g_{\mu\nu} \neq g^{\mu\nu}$, but we won't get into that.

Now let's look again at Einstein's equation:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda \eta_{\mu\nu}.$$

We see that since μ and ν can each take any value from 0 to 3, this is actually 16 equations. There are some symmetries that mean that not all 16 equations are independent: basically $G_{\mu\nu}$, $T_{\mu\nu}$, and $\eta_{\mu\nu}$ are all symmetric, meaning $T_{02} = T_{20}$, etc. which makes 6 of the equations redundant, thereby reducing the number to 10. We also see the $\eta_{\mu\nu}$ flat space metric. The constant term Λ multiplying $\eta_{\mu\nu}$ is called the **cosmological constant** and was introduced by Einstein under the false belief it would stop the Universe from expanding. We will come back to this term after looking at the term with the rank 2 $T_{\mu\nu}$ tensor. The other factors in this term are just Newton's constant, G, and the speed of light. The two-index $T_{\mu\nu}$ is called the **Stress-Energy tensor**. This is where you specify what the gravitational sources are: the masses, energies, stresses, etc. that curve spacetime. We have said that in GR, mass and energy curves spacetime, and we specify exactly what those source terms are in this tensor. Note for a black hole the source term would be a delta-function at the origin with a mass, m, while for the expanding Universe case it would be a uniform density of energy throughout the Universe. A fairly general case often considered is the perfect isotropic fluid. In this case $T_{00} = -\rho(t, r, \theta, \phi)$, $T_{ii} = p(t, r, \theta, \phi)$, and for $\mu \neq \nu$, $T_{\mu\nu} = 0$, where ρ is the energy density and p is the pressure. Note that we are using the convention that **greek indices such as μ and ν run from 0 to 3, while latin indices such as i, and j run only from 1 to 3**, that is, latin indicate the space-like dimensions.

It would take quite a bit of work to understand why this is the form of the stress energy tensor, but some idea can be had by thinking in terms of 4-vectors. Time and space are combined into a 4-vector because as one speeds up, they can be transformed in a well known way (by the Lorentz transformation). What is constant during such a "boost" is neither time nor the space but the contraction of the 4-vector, which is called the invariant interval s (or τ). Likewise it is the contraction $p_\mu p^\mu = m^2$ which is constant in the momentum 4-vector. So somehow the source of the gravitational field (curvature) must be 4-vector like. We know that mass, that is mass per unit volume, or density, causes gravity, but mass is really made of energy and momentum. Thus both energy and momentum are sources of gravity. But we need the densities, that is the mass or momentum per unit space (or time). This is because GR is a local theory described by differential equations, so it is only the amount of something at a local point (the density) that comes into the equations. The energy per unit volume is the density and is written ρ above. It can be a function of space and time, but it is an initial condition that is specified before solving the field equations. Since Energy is in the zero position in a 4-vector, the density goes into the zero-zero position of $T_{\mu\nu}$. In the momentum position we put some kind of density of momentum what ever that is. In fact, in GR it is the fluxes of momentum that go in the T_{ii} positions. The flux of momentum is another way of saying the pressure, $\tilde{p}$ (For pressure I put a tilde over the $\tilde{p}$ to distinguish it from momentum, which I just call p). This can be understood by thinking of what pressure is. Pressure is force per unit area: $\tilde{p} = F/A$, and $F = dp/dt$, is the change in momentum, so $\tilde{p} = dp/(dtdA)$, or pressure equals the flux of momentum through a given small surface element. This is a more general definition of pressure than you may be used to, but is great because it works even when the particles causing the pressure are not bouncing off walls. In the case where particles are bouncing off walls, it gives the same number as the normal defintion of pressure.

In general a component of the stress-energy tensor $T_{\mu\nu}$ is the flux of the μ component of momentum

(p^μ) across a surface of constant x^ν. So again T^{00} is the flux of $p^0 = E$ across a surface of constant $x^0 = t$, that is the energy density $\rho(x, y, z)$, and T^{11} is the flux of momentum $p^1 = p_x$ across a surface of constant $x^1 = x$, or the x-momentum per time in the y-z direction, that is, the pressure.

For the isotropic fluid case (which covers many cases of interest) the stress-energy Tensor then reads:

$$T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

For non-isotropic cases, we have to specify the off-diagonal terms also. Keeping with the 4-vector idea, these involve energy and momentum fluxes in space and time. Thus these are stresses, e.g. momentum fluxes in the y-direction transfered across the x-direction, i.e. y-forces across x-surfaces. It is surprising, but inevitable that just squeezing a ball gives rise to gravitational fields!

As a little aside, we note that it is energy, not just rest mass, that goes in the stress-energy tensor, so electric and magnetic fields also count. Thus electromagetic fields and radiation give rise to curved spacetime and "gravitational fields". It is a little complicated, but for a pure electromagnetic field one can write out the stress energy tensor in terms of electric field E and magnetic field B as follows:

$$T^{\alpha\beta} = F^\alpha_\gamma F^{\beta\gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta},$$

where we raise and lower using the metric $g_{\mu\nu}$, and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Throughout we use the Einstein summation convention, so there are many sums in the above equations. In electromagnetic theory, the tensor $F^{\mu\nu}$ is called the field strength tensor, and is related to the 4-vector form of the electric and magnetic fields, A^μ. The above tensor notation is quite fun in that it allows Maxwells' equations to be written in a very simple form, which we don't reproduce here.

Now look again at the 3rd term in the Einstein equations. We now see that since Λ is a constant it represents a uniform stress-energy. In fact, it represents an energy density of empty space; that is, remove all matter, radiation, etc. of every sort from the space and Λ represents the energy density then. This seems like a weird concept, and when the expansion of the Universe was discovered, Einstein decided that the cosmological constant term was not needed. In fact, this term is mathematically allowed, and actually is completely acceptable in modern quantum field theory, and so should be included. It is an experimental question what the value of Λ is. If there is no energy density of the vacuum, then one can just set $\Lambda = 0$. When we write $T_{\mu\nu}$ for a black hole we typically don't include Λ and so set $\Lambda = 0$. If we didn't set it to zero then we would not get the Schwarzschild metric, but a metric that actually has two horizons in it, one near the normal Schwarzschild radius, and another at a very great distance corresponding to the eventual "edge" of the visible Universe. Maybe more on that next quarter in Phyiscs 162!

Next look at the first term in Einstein's field equations: $G_{\mu\nu}$ which is called the Einstein tensor. We to define need one more notational convention: We will write derivatives with respect to x^μ, as ∂_μ. That is $\partial_\alpha f(x) = \partial f/\partial x^\alpha$, and $\partial^\alpha f(x) = \partial f/\partial x_\alpha$, where as usual if $\alpha = 0$, this means a time derivative, and if $\alpha = 2$, is means a derivative with respect to y, etc.

Now the Einstein tensor is defined in terms of the metric and contractions of the **Reimann curvature tensor** $R^\lambda_{\mu\nu\kappa}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

where R is called the **Ricci scalar curvature**

$$R = g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa}.$$

Don't forget that we are using the Einstein summation convention so there are actually 256 terms in the equation for R! We raise and lower indices of the Reimann and other tensors by contracting with the metric, for example: $R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^\sigma_{\mu\nu\kappa}$. The **Ricci tensor** is formed by

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa}.$$

Note that the rank four Reimann tensor is a shorthand way of writing 256 numbers! It is formed from the metric and the rank three affine connection $\Gamma^\lambda_{\mu\nu}$ which contains 64 numbers:

$$R^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta}.$$

The final definition is the **affine connection** also known as the **Christoffel symbol**, which involves derivatives of the metric:

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right).$$

Whew! So starting with the the stress energy tensor and the cosmological constant, one solves this set of coupled differential equations for the metric $g_{\mu\nu}$. That is, we need to find what sixteen quantities contained in $g_{\mu\nu}$ can be differentiated and summed over as specified above so as to satify the Field equations. Given how complicated the above formulas are you might guess that this is not an easy task! In fact it is made even more difficult because the solution of the above equations is not unique. There are certain contraints call the **Bianchi Indenties** that are simultaneously satisified

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} = 0,$$

where for any tensor V, the co-variant derivative $V^\mu_{;\nu}$ is defined

$$V^\mu_{;\nu} = \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa} V^\kappa.$$

The Bianchi identities are where convervation of momentum and energy are encoded into General Relativity. These identities plus the symmetric nature of the tensors, imply extra symmetries in the field equations that reduce the 16 differential equations to effectively only 6, but one must be very careful in consistently picking values of some seemingly arbitrary functions caused by these coordinate invariances. This is called choosing a gauge. The results you get may depend on the value of these arbitrary functions (the gauge you choose), and thus workers who picked different values might get different answers. Luckily, any quantity that can be actually measured in an experiment must be independent of the gauge and therefore the same in every gauge. It is important in GR to calculate things that can be measured.

One can thus see why solving these equations is not easy. For realistic conditions (realistic values of the stress-energy tensor) there are only a few known analytic solutions. Even on a computer these equations are extremely difficult and only recently has progress been made solving them for realistic situations. And when you solve them what you get is the metric. Thus, while there is some value in going through the steps of finding the analytic solution for the black hole stress-energy, in the end, to find the physics you still just start from the metric as we did in this class. Now you see why in this class we decided to skip the field equations and just pull metrics out of a hat.

Hopefully this lecture helped you see some of the full complexity of GR.