Strategy in Repeated Losing Games

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We study the strategy in repeated losing games. For the continuous limit, we propose two wagering strategies which we find have the highest overall probability to win the game. In the discrete case, one of the strategies is not applicable. So we propose a hybrid strategy instead. We numerically compute the probability of winning the game under different conditions. We find it is possible for the player to win the game given large enough initial balance. We also compute the average wager and the expected number of trials with different strategies, and analyze the advantages of those strategies.

INTRODUCTION

Many games in real life can be described by repeated games. We explore the strategies in such kind of games where the player has identical probability (smaller than 0.5) to win in each round. The game is initially inspired by an online challenge[1], which can be described as follows. The player is playing a 2-player game with a casino. The player start with a balance of x_0, while the casino begins with $N - x_0$, where $0 < x_0 < N$. The player's probability of winning each round is a constant value $0 < p < 0.5$. At each round r the player may wager any amount b_r which the casino must match. Neither one may have a negative balance at any time, which means b_r should satisfy both $b_r \leq x_r$ and $b_r \leq N - x_r$. The winner of each round take the pot ($2b_r$). The game ends as soon as the balance of either one reaches zero ($x_r = 0$ or $x_r = N$). The only thing that the player can control in this game, is the wager in each round. A wagering strategy is then a mapping from all possible balance to the next wager ($X \Rightarrow B$).

For any strategy, let $q(x_0)$ be the probability that a player eventually beats the casino given initial balance of x_0. Since we don't have any further information about the initial balance, we assume it's uniformly distributed, and use the average winning probability over all available initial balance $Q = \frac{1}{|X|} \sum_{x_0 \in X} q(x_0)$ as the metric to evaluate strategies.

This game can also be described by a biased random walk, in which the probability that the walker moves towards each direction is fixed, while the walker can choose the length of next step based on its current location.

Since the probability to win in each round is smaller than 0.5, intuitively the overall probability to win the game (Q) should also be smaller than 0.5 (that's how casino makes money). What we can do is to find the best strategy which maximize Q to help the poor player. We will start from the continuous limit, which is (was) always preferred by physicists. Then we will move on to the general case where N can be any natural number. We propose several strategies and compute the probability of winning the game. Finally we will compare these strategies by their average wager and number of trials.

CONTINUOUS LIMIT

In the limit where N is very large, we can regard the game as a continuous random walk, where the steps are arbitrary positive real number rather than integer. Without losing generality, we set $N = 1$, while balances x and wagers b are real numbers between 0 and 1.

Make It or Break It: An Intuitive Strategy

Intuitively speaking, since it is more likely to lose than to win in a single round, the more rounds we play, the smaller chance we will win. In order to minimize the number of rounds, a simple method is to increase the wagers to their maximal possible values so that the game will end as early as possible. Thus, we come up with a naive strategy:

$$b(x) = \min\{x, 1 - x\}, \tag{1}$$

as shown in figure 1. We would call this strategy *make it or break it*, since the player only need one round to win when $x \geq 0.5$, or to lose when $x \leq 0.5$. We can calculate Q for this strategy by writing down the equation:

$$Q = \frac{1}{2}p + \left[\frac{1}{2}(1 - p) + \frac{1}{2}p\right] Q. \tag{2}$$

The first term in the RHS is contributed by the lucky players who start with $x_0 > 0.5$ and win the first round. The second term comes from the fact that uniformly distributed players with $x_0 > 0.5$ and lose the first round are again uniformly distributed with balance $0 < x < 1$, and so are the players with $x_0 < 0.5$ and win the first round. So, the distribution of these two parts of players is the same as the beginning of the game. Since the game is a homogeneous Markov process, the overall chance of those two parts of players to win is again Q, thus leads to the second term in the RHS. Using Eq. 2, we immediately get $Q = p$. As you will see, this intuitive strategy is one of the most effective wagering strategies.

Never Draw Back: The Beauty of Fractal and Self-Similarity

Inspired by the fascinating fractal, we find a self-similar strategy which has identical overall probability to win as the *make it or break it* strategy. (Actually it's inspired by the results from brute-force search in the discrete case, but I will never tell you the truth in my report.) To describe the strategy, we first need to binary expand any real number x between 0 and 1:

$$x = \sum_{k=1}^{\infty} \frac{c_k(x)}{2^k}, \tag{3}$$

where $c_k(x) = 0$ or 1 is the kth binary expansion coefficient of x. Let

$$l(x) = \max\{k | c_k(x) = 1\} \tag{4}$$

be the largest order of binary expansion with nonzero coefficient, then the *never draw back* strategy can be expressed as

$$b(x) = \frac{1}{2^{l(x)}}, \tag{5}$$

as shown in figure 2. Notice that $b(x) > 0$ only for the rational numbers, since for any irrational number x we have $l(x) = \infty$, thus the Lebesgue measure of the set of balances that have nonzero wagers is zero. This strategy is similar to Thomae's function, which means the strategy is discontinuous at rational numbers. It is worth noticing that in this strategy, the wagers that the player choose will always increase monotonically, at least with a factor of 2. That is why we call this strategy never draw back.

The *never draw back* strategy is self-similar, which makes it easy to compute the overall probability to win Q. With the self-similarity, we know the probability that the balance finally reaches 0 and 0.5 given initial balance uniformly distributed in range $(0, 0.5)$ is $1 - Q$ and Q, respectively. Similar result holds for initial balance greater than 0.5. Once the

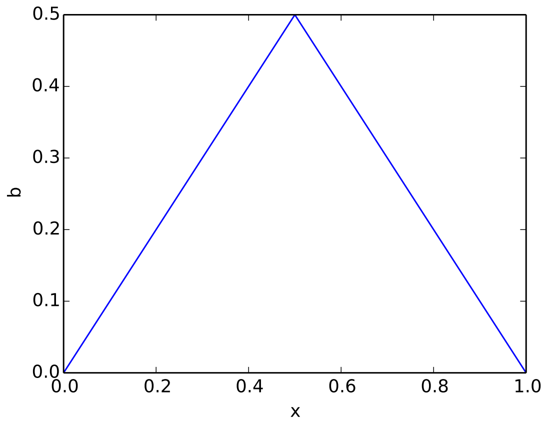

FIG. 1. Next wager (b) as a function of current balance (x) for the *make it or break it* strategy.

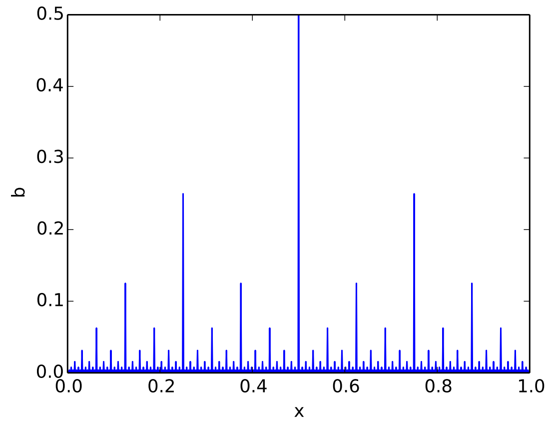

FIG. 2. Next wager (b) as a function of current balance (x) for the *never draw back* strategy.

balance is exactly 0.5, the game will end in one round, with the probability to win equals p. Thus, we can write down the equation for Q:

$$Q = \frac{1}{2}Q + \left[\frac{1}{2}(1-Q) + \frac{1}{2}Q\right] p. \tag{6}$$

From Eq. 6 we can get $Q = p$, which is exactly the same as previous *make it or break it* strategy.

Is There a Better Strategy?

We have already found two strategies that have the overall probability to win $Q = p$. Since our goal is to maximize Q, we want to know if there are other strategies with $Q > p$. Actually, according to the results from brute-force search in the discrete case, it seems that the two strategies we find have the highest possible Q. Here we want to prove that the upper bound of Q is $Q_{max} = p$. Actually we already proved $Q_{max} \geq p$ by constructing the strategies above, but we don't manage to prove $Q_{max} \leq p$. So we leave it as our conjecture.

DISCRETE CASE

After a glance at the beautiful continuous limit, we have to go through the discrete case, since money is always discrete in reality. Everyone will be happy if the condition $N \to \infty$ will lead to the continuous limit. However, this is not correct here, since some strategies heavily rely on the wager at some single balances. As a simple example, we cannot use the *never draw back* strategy when N is an odd number, because there is no central balance where the wager should be equal to $N/2$. One may argue that when N is large enough, slightly changing the strategy (such as change the wager at only one balance) will not have great impact on Q. Unfortunately, in the strategies where single balances serve as important hubs (such as the central point in the *never draw back* strategy), change of the wager at single balance will significantly affect the value of Q. For instance, in the discrete *never draw back* strategy where $N = 1024$ and $p = 0.37$, changing the central wager from 512 to 1 will reduce Q from 0.37 to 0.187. As a comparison, this will not happen in the *make it or break it* strategy, where changing the wager at any single balance will not result in observable changes in Q.

Hybrid Strategy

It is straightforward to apply the *make it or break it* strategy to the discrete case simply by choosing the maximal wager in each round. However, it is only possible to use the *never draw back* strategy completely when $\log_2 N$ is an integer. In order to use the *never draw back* strategy for any N, we develop the *hybrid strategy*, which can be described recursively. The *hybrid strategy* in range $(N_0, N_0 + N)$ is: if N is odd, then use the *make it or break it* strategy in range $(N_0, N_0 + N)$; if N is even, then set $b(N_0 + N/2) = N/2$ and use the *hybrid strategy* in range $(N_0, N_0 + N/2)$ and $(N_0 + N/2, N_0 + N)$. One can quickly check that if N is odd, then the *hybrid strategy* is just *make it or break it* strategy; if $\log_2 N$ is an integer, then the *hybrid strategy* is just *never draw back* strategy. For arbitrary N, the *hybrid strategy*, as indicated by its name, is a mixture of both strategies: the *never draw back* strategy is used until the balance is divided into several identical intervals of which the length is odd number; then the *make it or break it* strategy is applied to each interval. We plot the *hybrid strategy* for $N = 1000$, as shown in figure 3. Wee see that the *hybrid strategy* loses its beautiful self-similarity, but obtains the continuousness (at most of the points).

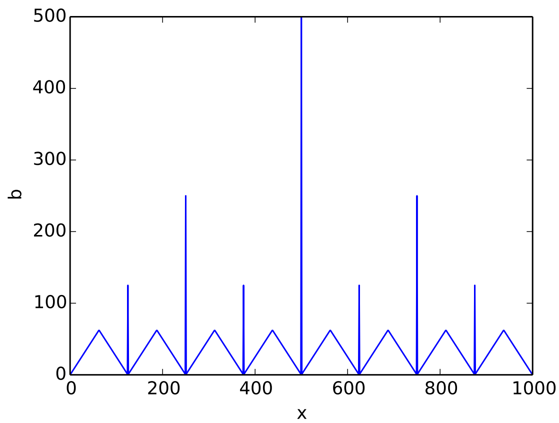

FIG. 3. Next wager (b) as a function of current balance (x) for the *hybrid strategy* with $N = 1000$. We see that the *hybrid strategy* consists of a *never draw back* strategy and several *make it or break it* strategies. If N is odd, the *hybrid strategy* reduces to *make it or break it* strategy; if $\log_2 N$ is an integer, then the *hybrid strategy* reduces to discrete *never draw back* strategy.

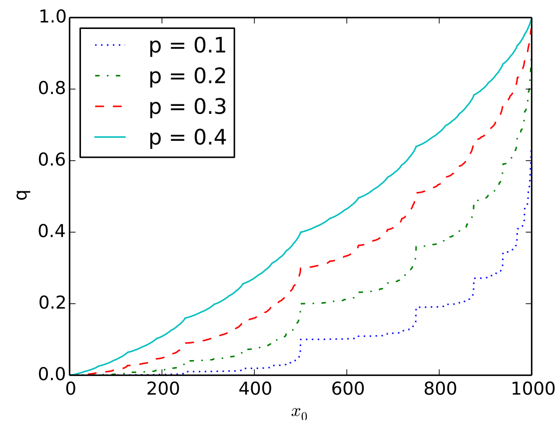

FIG. 4. Probability of winning the game with $N = 1000$ and different p. Notice that the *make it or break it* strategy and the *hybrid strategy* always have exactly the same probability of winning given any initial balance. The probability of winning in the continuous limit can be obtained by letting $N \to \infty$, regardless the parity of N.

Probability of Winning

As shown in previous section, in the continuous limit, we have $Q = p$ for both *make it or break it* strategy and *never draw back* strategy. Using the same method, it is straightforward to show that any *hybrid strategy* also has $Q = p$ if the balances in the *make it or break it* part become continuous. However, sometimes the player also want to know the probability of winning the game given any initial balance to decide whether he should play or not. We numerically calculate the probability of winning for both *make it or break it* strategy and *hybrid strategy*. An example where $N = 1000$ is shown in figure 4. From the numerical results, we find that the *make it or break it* strategy and the *hybrid strategy* have exactly the same probability of winning given any initial balance x_0 and total balance N. That means these two strategies are identical, even in the continuous limit, if we only care about the probability to win. Moreover, if we choose $N = 2^M$ and let $M \to \infty$, we then know that in the continuous limit, the *make it or break it* strategy and the *never draw back* strategy also have the same probability to win given any initial balance, which is consistent with the previous result which says they have the same Q.

From figure 4 we can see that the probability of winning in these strategies have some self-similarity, even if the strategies themselves may not have such property. To be more specific, the probability of winning has large "jumps" (although still continuous) at the points where the *never draw back* strategy has large wagers. These "jumps" are significant for small p, since the left derivative of the probability with respect to initial balance at largest initial balance will diverge when $p \to 0$. As an example, from figure 4 we know when $p = 0.1$, the probability of winning will decrease 50% if the initial balance is decreased by only a few dollars from \$500, while the probability will almost remain the same if the initial balance is increase by \$100 from \$500. Thus, the player should be careful about initial balance, since the probability of winning the game is very sensitive to initial balance at some certain points.

How to Win The Game?

One of the most important question in this problem is, how to beat the casino given $p < 0.5$. From figure 4 we can see that the player can win the game only when his initial balance is larger than the initial balance of casino. Figure 5 shows the initial balance needed to win as a function of p. When $p < 0.1$, it is very hard for the player to win since he must have much much more initial balance than the casino. When $p > 0.1$, the initial balance needed to win decrease like a linear function. Notice that the strategies we proposed share the same probability of winning and the initial balance to win. The trade-off of different strategies will be analyzed in the following parts.

Average Wager

As we shown above, the *make it or break it* strategy and the *hybrid strategy* seem to be identical because they share exactly the same probability of winning given any initial balance. However, the *hybrid strategy* is preferred if we introduce another metric other than the overall probability of winning (Q). According to the original online challenge problem, when two strategies have the same Q, we compare the average wager over all initial balance

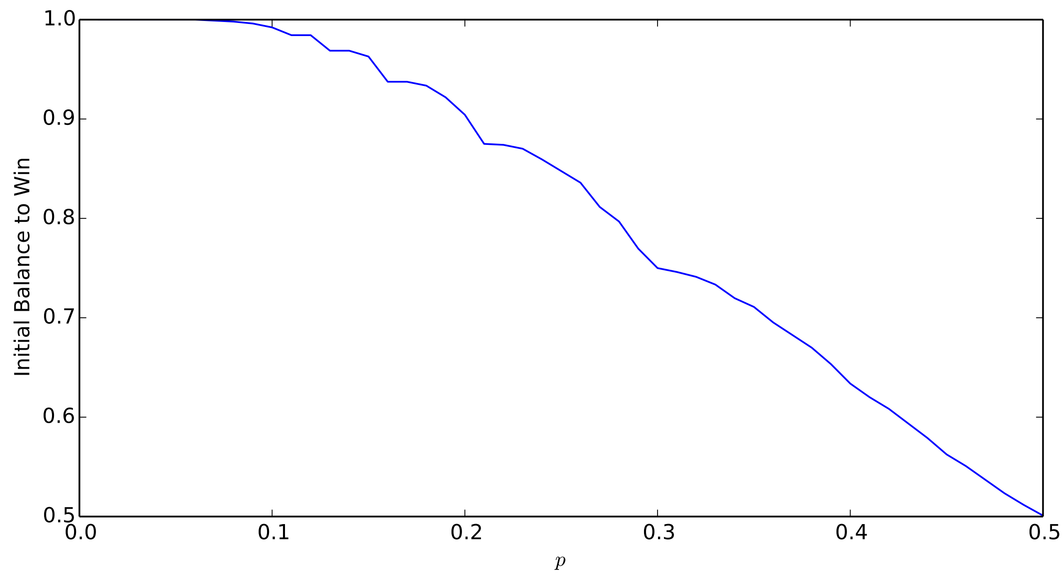

FIG. 5. Initial balance needed to win the game with respect to p. The balance is rescaled so that total balance $N = 1$.

which is defined as $W = \frac{1}{|X|}\sum_{x_0 \in X} b(x_0)$. The strategy with lower W is preferred. Although there is no official explanation to this metric, we imply that the goal is to reduce capital flow, since the challenge is held by a trading company. When used in finance, smaller wager is preferred because the rest of the money can be used in other investment, thus increasing the fund utilization efficiency.

In the continuous limit, W can be defined by the integration of $b(x)$. One can easily get $W = \frac{1}{4}$ for the *make it or break it* strategy. For the *never draw back* strategy, we have $W = 0$ because $\sum_{k=0}^{\infty} \frac{k}{2^k} = 2$. Thus, from the aspect of reducing capital flow, the *never draw back* strategy is better than the *make it or break it* strategy, since the former one barely choose large wager. In the discrete case, it is then easy to see that the *hybrid strategy* has smaller W than the *make it or break it* strategy. So the *hybrid strategy* is preferred considering the average wager W.

Number of Trials

In such a game, it is naturally to ask how long does the game last. Due to the complexity of the strategies, we find it hard to calculate the expected number of trials theoretically.

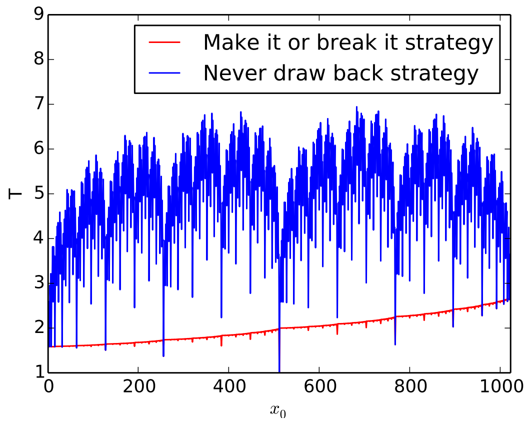

FIG. 6. The number of trials T as a function of initial balance. $N = 1024$ and $p = 0.37$ is used in this plot. When $N = 2^M \to \infty$, T will increase as $\log N$ for the *never draw back* strategy, but will remain stable for the *make it or break it* strategy.

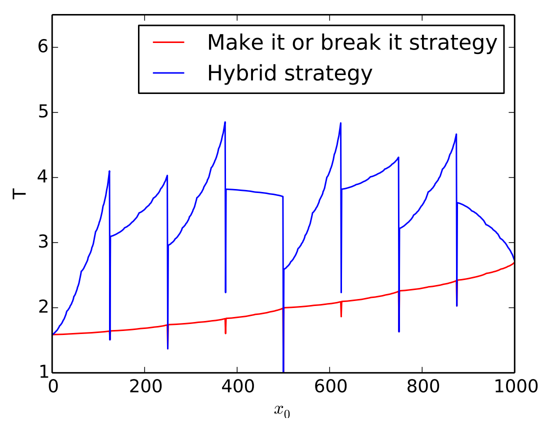

FIG. 7. The number of trials T as a function of initial balance. $N = 1000$ and $p = 0.37$ is used in this plot. The shape of the $T - x$ curve is sensitive to N for the *hybrid strategy*, since it will reduce to the *make it or break it* strategy or the *never draw back* strategy for some N.

Instead, we compute it numerically. Again, we calculate the expected number of trials in the continuous limit by choosing $N = 2^M$ and let $M \to \infty$. However, we find the number of trials of the *never draw back* strategy diverges when $M \to \infty$. To be more specific, it grows with the logarithm of N. This divergence will not happen in reality, because the logarithm of any amount of money in any unit in real life is not a big number at all. We choose $N = 2^{10} = 1024$ and plot the number of trials with different initial balances in figure 6. As shown in the plot, for the *never draw back* strategy the number of trials is discontinuous with respect to initial balance. Notice that even for the *make it or break it* strategy, the number of trials is discontinuous but will not diverge when $N = 2^M \to \infty$. In the continuous limit, it becomes discontinuous at any rational initial balance, just like the *never draw back* strategy shown in figure 2.

For arbitrary N where we use the *make it or break it* strategy or the *hybrid strategy*, the number of trials is discontinuous only at points where the wager of *hybrid strategy* is discontinuous (see figure 3). An example with $N = 1000$ and $p = 0.37$ is shown in figure 7. We see from the example that the number of trials in the *hybrid strategy* has really strange shape and cannot be well described by a simple function.

Just like the average probability of winning Q and the average wager W, we can use the average number of trials L as a metric of strategies. From previous examples, it is easy to see that the *make it or break it* strategy has smaller L than the *hybrid strategy*. So if we want to end the game as soon as possible, it may be better to use the *make it or break it* strategy.

CONCLUSION

For repeated losing games, we start from the continuous limit and propose the *make it or break it* strategy and the *never draw back* strategy which we find have the highest overall probability to win $Q = p$. In the general discrete case, we may not use the *never draw back* strategy. Instead, we propose the *hybrid strategy*. We find that the *make it or break it* strategy and the *hybrid strategy* share the same probability of winning the game given any initial balance. We find that it is possible for the player to win the game if his initial balance is much more than the casino. We also numerically compute the average wager and the expected number of trials with different strategies. We find the *hybrid strategy* has less average wager which is better for reducing capital flow, while the *make it or break it* strategy has smaller number of trials so the game will end sooner.

[1] "Second annual jump trading challenge," (2013).