

# Black-Scholes Equation for Option Pricing

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## 1. Introduction

In early 1970s, Black, Scholes and Merton achieved a major breakthrough in pricing of European stock options and their development became known as the Black-Scholes-Merton model later on which had a huge influence on the way traders price and hedge derivatives nowadays. First, we are going to discuss derivation of the equation, then estimation of volatility from historical data or implying it from option prices using the model. Eventually we'll come to a discussion of extension of the model to deal with European call and put options as well as American call ones.

### 1.1 Arbitrage and financial derivatives

One of the fundamental concepts underlying the theory of financial derivative pricing and hedging is that of arbitrage. This can be loosely stated as “there is now such thing as free lunch” or, more formally, in financial terms, there are never any opportunities to make an instantaneous risk-free profit. (Or, being more exact, they can't exist for some long time).

Within the Black-Scholes model we are going to consider risk-free investments that give a guaranteed return with no chance of default. The second underlying assumption is that the greatest risk-free return that one can make on a portfolio of assets we are going to make is the same as the return if the equivalent amount of cash were placed in a bank. It's important to distinguish those two kinds of financial objects- assets and options (which a portfolio is basically). The former is “real”, “tagible” like oil or gold or stocks while the latter is “imaginary”- it's an agreement on buying or selling assets over some stretch of time, or, in more refined terms- call and put options respectively. Assets price is assumed to follow random walk within the model and after constructing of a proper portfolio (i.e. getting rid of stochastic terms) it's price as a function of underlying

asset and time  $V(S, t)$  is going to obey Black-Scholes equation. After having the plan sketched, let's go through the process in details.

## 1.2 Lognormal distribution of stock prices

The model assumes that percentage changes in the stock price in a short time are normally distributed:

$$\frac{\Delta S}{S} \propto \phi(\mu \Delta t, \sigma^2 \Delta t)$$

Where  $\phi(m, v)$  denotes a normal distribution with mean  $m$  and variance  $v$ . The model implies that

$$\ln S_T - \ln S_0 \propto \phi[(\mu - \sigma^2/2)T, \sigma^2/T]$$

So that

$$\ln S_T - \ln S_0 \propto \phi[(\mu - \sigma^2/2)T, \sigma^2/T]$$

and

$$\ln S_T \propto \phi[\ln S_0 + (\mu - \sigma^2/2)T, \sigma^2/T]$$

where  $S_T$  is the stock price at a future time  $T$  and  $S_0$  is the stock price at times  $t$ .  $S_0$  is the stock price at time 0. This equation shows that  $\ln S_T$  is normally distributed, so that  $S_T$  has a lognormal distribution. It can be shown that the expected value  $E(S_T)$  of  $S_T$  is given by

$$E(S_T) = S_0 e^{\mu T}$$

## 1.3 Option values, payoffs, and strategies

Now let's turn to option pricing. Let's consider what happens at the moment of a call option, that is, at time  $T$ . If  $S > E$ , at expiry time  $T$ , it makes financial sense to exercise option, having over an amount  $E$ , to obtain asset worth  $S$ . The profit from such a transaction is then  $S - E$ . On the other hand, if  $S < E$  at expiry, since we would make a

loss of  $E - S$ . In this case the option expires worthless. Thus, the value of the call option at expiry can be written as

$$C(S, T) = \max(S - E, 0)$$

As we get nearer to expiry date we can expect option to approach  $C(S, T)$ .

Similar analysis can be made for put options. If the exercise price is lower than the value of underlying assets at expiry, holder of the put option would not sell the asset, so the value of this put option is 0. As a whole, value of put option at expiry is

$$P(S, T) = \max(E - S, 0)$$

## 2. Derivation of Black-Scholes equation

If we assume that option value and corresponding stock value are perfectly correlated, then we can eliminate risk, or in other words random fluctuation in the price, by investing on a portfolio of right amount of options and stocks. So that the return from this portfolio should be equal to risk free interest rate as a result of supply and demand equilibrium, also to avoid arbitrage. The underlying equation is called Black-Scholes equation.

This leads to an arrangement of the following sections. Above all, we need, if there exists any, a mathematical foundation Ito's lemma to kick off everything. Then, as a profound application, Ito's lemma is used to determine stock price changing, where we can get a sense of the randomness of stocks. Finally, we try to eliminate the random term thus arrive at Black-Scholes equation for option price.

### 2.1 Ito's Lemma

Before starting with the price of stocks, which follows a random walk with drift, it is useful to introduce Ito's lemma, which tells the stochastic behavior of a function  $G(x, t)$  where  $x$  is conducting a random walk with drift.

Given

$$dx = a(x, t)dt + b(x, t)dz, \quad (2.1)$$

where  $z$  follows a Wiener process,  $dz \sim \sqrt{dt}$ .  $a(x, t)$  is the drift rate, and  $b(x, t)$  the variance rate. Keeping terms up to order of  $dt$ ,

$$dG(x, t) = dG(x(z, t), t) = \frac{\partial G}{\partial z} dz + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial z^2} (dz)^2 + \dots \quad (2.2)$$

Using definition of this process, we can get the following derivatives

$$\frac{\partial G}{\partial z} = \frac{\partial G}{\partial x} \frac{\partial x}{\partial z} = \frac{\partial G}{\partial x} b;$$

$$\frac{\partial^2 G}{\partial z^2} = \frac{\partial^2 G}{\partial x^2} b^2;$$

$$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} a.$$

Plugging them back into (2.2), and ignore higher order terms, we arrive at Ito's lemma

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz. \quad (2.3)$$

It means that  $G(x, t)$  has a drift velocity

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2;$$

and variance rate

$$\left( \frac{\partial G}{\partial x} b \right)^2.$$

## 2.2 Stock price

With Ito's lemma, we can derive evolution of stock price ('s distribution) mathematically.

Let's assume stock price  $S$  follows the following random walk

$$dS = \mu S dt + \sigma S dz, \quad (2.4)$$

Where  $\mu$  is the stock's expected rate of return, and  $\sigma^2$  is its variance rate.

Rather than the absolute price, it is the percentage change in the price that follows a normal distribution

$$\frac{dS}{S} = \mu dt + \sigma dz. \quad (2.5)$$

Now use Ito's lemma. Define

$$G = \ln S,$$

So that

$$\frac{\partial G}{\partial S} = \frac{1}{S}; \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}; \frac{1}{S}.$$

From the assumption

$$a = \mu S; b = \sigma S.$$

Plug all of them into Ito's lemma, we get

$$d(\ln S) = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz. \quad (2.6)$$

So  $\ln S$  has a drift rate  $\mu - \frac{1}{2} \sigma^2$ , and variance rate  $\sigma$ .

Further we can get from the above equation

$$\begin{aligned} \ln S_T - \ln S_0 &\sim \phi \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) T, \sigma^2 T \right], \\ \text{or } \ln S_T &\sim \phi \left[ \ln S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) T, \sigma^2 T \right]. \end{aligned} \quad (2.7)$$

$S_T$  is the stock price at time T, and  $\ln S_T$  is a normal distribution with drifting mean and variance.

Taking average over all possible prices at time T, we get the expected stock price at time T

$$S_T = S_0 e^{\mu T} \quad (2.8)$$

which means on average stock price is growing exponentially. Of course the growing rate, or drift rate  $\mu$  mathematically, depends on the performance of company and the economical environment of the market.

### 2.3 Black-Scholes equation

Changing in stock price  $S$  due to changing in time is

$$dS = \mu S dt + \sigma S dz$$

which is a random walk with drift  $a = \mu S, b = \sigma S$ .

Price of option based on this stock is a function of  $S$ , and time  $t$  of course, denoting  $V(S, t)$ . According to Ito's lemma,  $V$  should change along with time in this way

$$dV = \left( \frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dz. \quad (2.9)$$

Staring at the above two equations for two seconds, we can happily eliminate the “dz” terms through multiplying first equation with  $\frac{\partial V}{\partial S}$  and then subtracting one from the other.

Financially we make a portfolio that short one share of option and long  $\frac{\partial V}{\partial S}$  shares of stocks. The value of this portfolio

$$\Pi = -V + \frac{\partial V}{\partial S} S,$$

And changing in this value

$$d\Pi = -dV + \frac{\partial V}{\partial S} dS. \quad (2.10)$$

Substituting the changes in stock price and option price, we get

$$d\Pi = - \left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \mu^2 S^2 \right) dt. \quad (2.11)$$

Since there is no randomness in our portfolio, its return rate should be exactly risk free interest rate  $r$  to avoid possibility of arbitrage.

So

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt = r\Pi dt. \quad (2.12)$$

Plugging in the value of portfolio, we get Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV. \quad (2.13)$$

### 3. Boundary and final conditions and solutions

Generally we can separate variables and get two equations of time and stock price respectively. The time equation is first order while the stock price equation is second order. So we need two boundary conditions at fixed prices. At exercise time, option price as a function of stock price is easy to know with certainty, called final conditions.

a) For European call options, at exercise time T

$$C(S, T) = \max(S - E, 0) \quad (2.14)$$

Where C is the value/price of this call option and E the exercise price.

When the stock price S is ever 0, its change due to time  $dS = 0$ , so S will remain 0 thus the call option is worthless. So

$$C(0, t) = 0. \quad (2.15)$$

The other boundary condition can be found at  $S \rightarrow \infty$

$$C(\infty, t) \sim S. \quad (2.16)$$

b) For a put option, final condition is

$$P(S, T) = \max(E - S, 0). \quad (2.17)$$

And when  $S = 0$ , the value of put at any given time t should reach E at expiry time T under risk free interest rate, which requires

$$P(0, t) = Ee^{-r(T-t)}. \quad (2.18)$$

If  $S \rightarrow \infty$ , then this put will never be exercised so

$$C(\infty, t) = 0. \quad (2.19)$$

If risk free interest  $r$  and variance rate  $\sigma^2$  are constant, explicit solutions can be achieved under the above boundary and final conditions for European call and put options, which are

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \quad (2.20)$$

And

$$P(S, t) = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2). \quad (2.21)$$

$N$  is the cumulative distribution function for a standardized normal random variable

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}x'^2} dx'. \quad (2.22)$$

Upper bounds are given as

$$d_1 = \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad (2.23)$$

and

$$d_2 = \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \quad (2.24)$$

Such solutions are plotted in Wilmott, Paul "*The Mathematics of Financial Derivatives*".

Let's take a look.

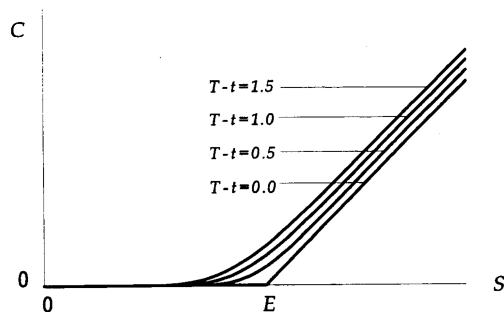


Figure 3.6 The European call value  $C(S, t)$  as a function of  $S$  for several values of time to expiry;  $r = 0.1$ ,  $\sigma = 0.2$ ,  $E = 1$  and  $T - t = 0, 0.5, 1.0$  and  $1.5$ .

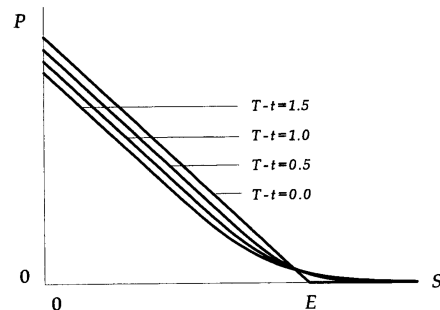


Figure 3.7 The European put value  $P(S, t)$  as a function of  $S$  for several values of time to expiry;  $r = 0.1$ ,  $\sigma = 0.2$ ,  $E = 1$  and  $T - t = 0, 0.5, 1.0$  and  $1.5$ .

As we can see, as time approaching expiry time, the prices approach payoff functions of options. If we take a closer look at the price evolution at fixed stock price, we would hopefully find an exponential increase or decrease.



## 4. Conclusion

First of all, recall how we eliminated randomness in the random walk of stock price and the change in portfolio value. The condition is that we must hold or sell  $\frac{\partial V}{\partial S}$  shares of stock at every moment. It requires frequent trading actions in stocks. And such trading actions should take into account current stock price and option price. This requirement arises two problems. One is the ignorance of transaction fee. The other is how to acquire real time data and calculate relevant prices fast. The latter one goes to computer scientists. But we can fix the first problem through adding friction to the random walk.

Second, Black-Scholes equation is a nonlinear second order equation. Generally, it is hard to find stable solutions. Hull's book shows a method named "Risk-neutral Valuation". Basically, it means in risk-neutral world the return rate is always risk free interest rate, and we can always get the payoff in real world through discounting the expected payoff in risk-neutral world. This provides a more achievable way to get solution. And the validity of this method lies on Black-Scholes equation's independence of risk preference, which means that people with any risk preference will get prices differing only by discounting.