Small Oscillations

Physics 200A

Section : Oscillations and Waves

→ Small Oscillations (Few N)

→ Chains, etc. N large

→ Continuum $N \to \infty$

Point: - normal modes
i.e. - Natural frequencies, resonances
- eigenfunctions

- energy - momentum propagation.

- symmetry is a key!

Then:

→ parametric instability

→ ponderomotive force.

Physics 200A

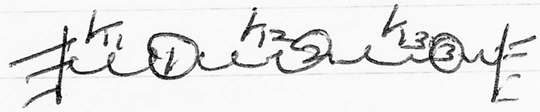
II.) Linear Oscillators and Continua

→ See L&L chapt. 23, 24
FW chapt. 4

.) Small Oscillations – Linear

→ Consider degrees of freedom / g.c. $q_1, q_2 \cdots q_i \cdots q_N$
s/t

$$U = U(q_1, q_2 \cdots q_N)$$



then, can expand near eqbrm. points q_{j0}, s/t

$$U = U_0 + \sum_j (q_j - q_{j,0}) \frac{\partial U}{\partial q_j}\bigg|_{q_{j0}}$$

$$+ \frac{1}{2} \sum_{j,k} (q_j - q_{j,0})(q_k - q_{k,0}) \frac{\partial^2 U}{\partial q_j q_k}\bigg|_{q_{j0}, q_{k,0}}$$

for minimum ⇒ $\begin{cases} \dfrac{\partial U}{\partial q_j}\bigg|_{q_{j0}} = 0 \\ \det \left| \dfrac{\partial^2 U}{\partial q_j q_k}\bigg|_0 \right| > 0 \end{cases}$

$$U = U_0 + \frac{1}{2} \sum_{jk} (q_j - q_{j,0})(q_k - q_{k,0}) \frac{\partial^2 U}{\partial q_j q_k}\bigg|_0$$

$$\equiv U_0 + \frac{1}{2} \sum_{jk} x_j x_k k_{jk}$$

↳ stiffness matrix

similarly $$T = \frac{1}{2}\sum_{jk} \overset{\text{mass matrix}}{\overset{\downarrow}{m_{jk}}}\, \dot{x}_j \dot{x}_k$$

$$(\det m_{jk} > 0)$$

$\Rightarrow$

$$L = \frac{1}{2}\sum_{jk}\left(m_{jk}\dot{x}_j\dot{x}_k - k_{jk}x_j x_k\right) \quad (*)$$

↳ generic Lagrangian

$\therefore$

$$\frac{d}{dt}\left(\sum_k m_{jk}\dot{x}_k\right) + \sum_k k_{jk}x_k = 0 \quad (*)$$

$$\Rightarrow \sum_k \left(m_{jk}\ddot{x}_k + k_{jk}x_k\right) = 0$$

thus $$\boxed{\sum_k \left\{ \underbrace{\underset{jk}{\ddot{x}_k}}_{\text{dimensionless}} + \underbrace{\left(\frac{k}{m}\right)_{jk}}_{\to\ \omega^2_{jk}} x_k \right\} = 0} \quad (*)$$

Eqn. Motion

N.B.:
$k_{jk} = k_{kj}$
$\lambda_{j,k} = \lambda_{kj}$ (*)

$$x_k = A_k e^{-i\omega t}$$

$$\Rightarrow \sum_k \left(-\omega^2 \underbrace{\lambda_{jk}}_{\substack{\text{mass matrix}\\ \text{normalized}}} + \underbrace{\omega^2_{j,k}}_{\text{frequency matrix}}\right)A_k = 0$$

eigenfrequencies : $\det\left|\omega^2_{jk} - \omega^2 I_{jk}\right| = 0$

$\Rightarrow$ collective mode frequencies

ratio amplitudes $\Rightarrow$ eigenvectors.

thus, solution $\rightarrow$ n eigenfrequencies ω_α^2
$\rightarrow$ n eigenvectors a_j^α

so, can write motion:

$$x_j = \sum_\alpha a_j^\alpha e^{-i\omega_\alpha t}$$

$\left\{ \begin{array}{l} j \rightarrow \text{comp.} \\ \alpha \rightarrow \text{eigenvalue label} \end{array} \right.$

$A_j^\alpha = C_\alpha a_j^\alpha$

i.e. eigenvector representation $\leftrightarrow$ orthonormal basis

Pf. Consider 2 eigenvalues ω_s^2, ω_r^2

$$\Rightarrow \quad \omega_s^2 \sum_k \lambda_{jk} a_k^s = \sum_k \omega_{jk}^2 a_k^s \qquad (1)$$

$$\omega_r^2 \sum_j \lambda_{jk} a_j^r = \sum_j \omega_{kj}^2 a_j^r \qquad (2)$$

$$\sum_j \left\{ (1) \times a_j^r \right\} - \sum_k \left\{ (2) \times a_k^s \right\} \Rightarrow$$

$$\sum_{jk} \left\{ \omega_s^2 \lambda_{jk} a_j^r a_k^s - \omega_r^2 \lambda_{jk} a_j^r a_k^s \right\}$$

$$= \sum_{jk} \omega_{jk}^2 (a_k^s a_j^r - a_j^r a_k^s) = 0$$

$$\Rightarrow (\omega_s^2 - \omega_r^2) \sum_{kj} a_j^r a_k^s \lambda_{jk} = 0$$

$$\omega_s^2 \neq \omega_r^2 \Rightarrow \begin{cases} \sum_{jk} \lambda_{jk} a_j^r a_k^s = 0 \\ \text{orthogonality of eigenvectors.} \end{cases}$$

normalization $\Rightarrow \sum_j \lambda_{jj} a_j^2 = 1$

so have general orthonormality condition

$$\boxed{\sum_{j,k} \lambda_{jk} a_k^s a_j^r = \delta_{jk}} \quad (*)$$

Can express general oscillation in terms eigenvectors and time dependent amplitudes

$$x_j = \sum_\alpha a_j^\alpha \eta_\alpha(t)$$

↳ describes time ~~and phase~~ evolution ~~evolution~~ and phase i.e. amplitude.

orthogonality $\Rightarrow$

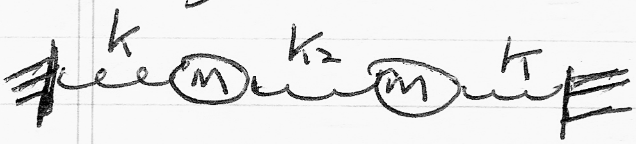
$$L = \sum_\alpha (\dot{\eta}_\alpha^2 - \omega_\alpha^2 \eta_\alpha^2)/2$$

$$\ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha = 0 \qquad \alpha = 1, \dots n$$

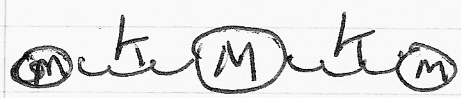
note: if det $| \omega_{ij}^2 - \lambda_{ij}\,\omega^2 | = 0$
has double root i.e. $\omega_\alpha^2 = \omega_\beta^2$

$\Rightarrow$ degeneracy! $\Rightarrow$ must arbitrarily introduce
some condition to
determine n orthog.
eigenvectors
(choice not unique)

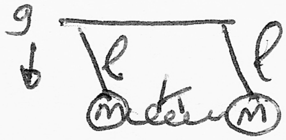
Best to proceed by considering series of
examples;

i) 

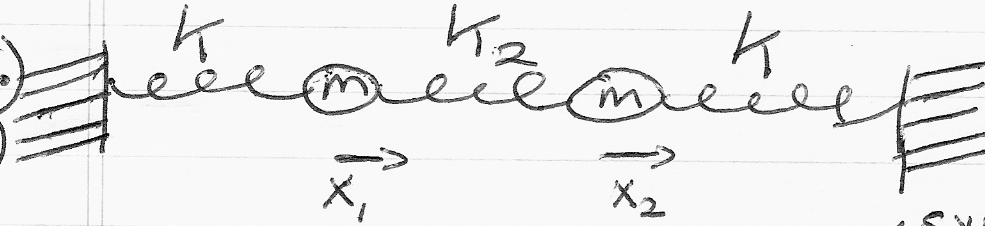
, $\begin{cases} V_{1,2} = -\alpha x y \\ \text{2 degs. freedom} \end{cases}$

ii) 

iii) molecular vibrations

i.e. triatomic molecule $\begin{cases} \text{- linear} \\ \text{- triangular} \\ \text{- asymmetric} \end{cases}$ sketch identical 

Ex. (i)



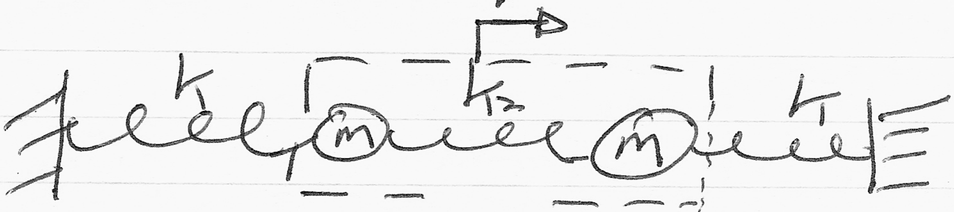
2 degs. freedom

symmetry ⇒ 2 normal modes < symmetric / antisymmetric

(lower energy ⇒ lower frequency)

symmetric ⇒ k_2 not extended

$x_1 = x_2$

i.e. 

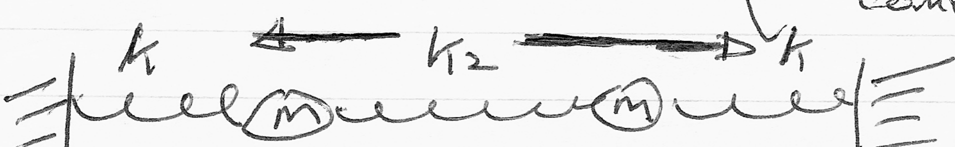
i.e. $M_{eff} = 2m$

$k_{eff} = 2k_1$

so $\omega^2 = 2k/2m = k/m$

antisymmetric ⇒ $x_1 = -x_2$

(higher energy ⇒ more spring compression)

i.e. 

↳ dbl extension

$$\Rightarrow m\ddot{x}_1 = -kx_1 - k_2(x_1 - x_2)$$

$$= -kx_1 - 2k_2 x_1$$

so $\omega^2 = \left(\frac{k_1}{m} + \frac{2k_2}{m}\right)$

Observe: − $k_2 \to 0$, 2 oscillators decouple
so coupling splits ω's

$$k/m \to \begin{cases} k/m + 2k_2/m & \left\{\begin{array}{l}\text{high freq,}\\ \text{antisymmetric}\end{array}\right. \\ k/m & \left\{\begin{array}{l}\text{low freq,}\\ \text{symmetric,}\end{array}\right. \text{ no } k_2 \text{ dep.}\end{cases}$$

− in general, antisymmetric → higher ω
(curvature energy) symmetric → lower ω.

− e.g.  $\omega^2 = 0$ symm. (trans.)
$\omega^2 = 2k/m$ anti-sym. (breather)

Cranking it out:

G.C.; x_1, x_2

$$L = \left[\frac{1}{2} m \dot{x}_1^2 + \frac{m\dot{x}_2^2}{2}\right] - \left[\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{k_2}{2}(x_2 - x_1)^2\right]$$

$\Rightarrow$

$$m\ddot{x}_1 = -k x_1 + k_2 (x_2 - x_1)$$

$$m\ddot{x}_2 = -k x_2 - k_2 (x_2 - x_1)$$

$\Rightarrow$

$$\ddot{x}_1 + \frac{k_1}{m} x_1 + \frac{k_2}{m}(x_1 - x_2) = 0$$

$$\ddot{x}_2 + \frac{k_1}{m} x_2 - \frac{k_2}{m}(x_1 - x_2) = 0$$

or better $\ddot{x}_1 + \omega_0^2 x_1 - k_2/m \; x_2 = 0$

$$\ddot{x}_2 + \omega_0^2 x_2 - \frac{k_2}{m} x_1 = 0$$

$\omega_0^2 = (k_1 + k_2)/m$

a) could just

- add $\Rightarrow$ $\ddot{x}_+ + \omega_0^2 x_+ - k_2/m \; x_+ = 0$

$$\ddot{x}_+ + k_1/m \; x_+ = 0$$

$\begin{cases} x_+ = x_1 + x_2 \\ \omega_+ = k_1/m \end{cases}$ $\eta_+ = x_+$

- subtract

$$\ddot{x}_- + \omega_0^2 x_- + \frac{k_2}{m} x_- = 0$$

$\begin{cases} x_- = x_1 - x_2 \\ \omega_- = (k + 2k_2)/m \end{cases}$ $\eta_- = x$

b) $x_1 = A e^{-i\omega t}$
$x_2 = B e^{-i\omega t}$

$$(\omega_0^2 - \omega^2) A - k_2/m \; B = 0$$
$$-k_2/m \; A + (\omega_0^2 - \omega^2) B = 0$$

$$(\omega_0^2 - \omega^2)^2 - (k_2/m)^2 = 0$$

$$\omega^2 = \omega_0^2 \pm k_2/m \quad \begin{cases} \omega^2 = k_1/m + 2k_2/m \\ \omega^2 = k_1/m \end{cases}$$

$A, B \Rightarrow$ eigenvectors.

$\omega^2 = \omega_0^2 + k_2/m$ $\qquad -k_2/m\ A - k_2/m\ B = 0$

$\qquad -k_2/m\ A - k_2/m\ B = 0$

anti

$A = -B$ so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$

$\omega^2 = \omega_0^2 - k_2/m$ $\qquad + k_2/m\ A - k_2/m\ B = 0$

$\qquad - k_2/m\ A + k_2/m\ B = 0$

symm

$A = B$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$ $\qquad$ high ω $\qquad$ low ω.

$\underline{\underline{\text{So}}}$ $\quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{C_1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_+ t} + \frac{C_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_- t}$

1b) $\quad V_{\underline{\underline{int}}} = -\alpha xy \quad \Rightarrow \quad$ anti sym $\rightarrow +\alpha + E_0$

$\qquad$ symm $\rightarrow -\alpha + E_0$

$\hookrightarrow$ interaction of two oscillators.

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2) + \alpha xy$$

$m\ddot{x} + kx - \alpha y = 0 \qquad \ddot{x} + \omega_0^2 x - \alpha/m\ y = 0$

$m\ddot{y} + kx - \alpha x = 0 \qquad \ddot{y} + \omega_0^2 y - \alpha/m\ x = 0$

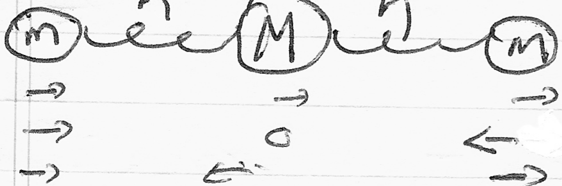
$\ddot{\eta}_+ + \omega_0^2 \eta_+ - \alpha/m\ \eta_+ = 0 \qquad \eta_\pm = x \pm y$

$\ddot{\eta}_- + \omega_0^2 \eta_- + \alpha/m\ \eta_- = 0$

$\left.\begin{array}{l} + \quad \omega^2 = \omega_0^2 - \alpha/m \\ - \quad \omega^2 = \omega_0^2 + \alpha/m \end{array}\right\}$ $\underline{\text{split}}$

symmetry ⟺ zero frequency modes
(why ⟺ displace with no change in energy) 10.

(c) k_1 k_1



$\omega^2 = 0$

$\omega^2 = k/m$ (more symm)

$\omega^2 = k\mu/mM$ (less symm)

Key point ⟹ no external forces, so CM constant ⟹ 1 degree symmetry.

⟹

$$m x_1 + M x_2 + m x_3 = \text{const} = 0$$

⟹ reduce $3\times3 \Rightarrow 2\times2$

$$L = \frac{1}{2}\left(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2\right) - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2$$

but $x_2 = -\frac{m}{M}(x_1 + x_3)$

⟹

$$L = \frac{1}{2}\left[m(\dot{x}_1^2 + \dot{x}_3^2) + M\frac{m^2}{M^2}(\dot{x}_1 + \dot{x}_3)^2\right]$$
$$-\frac{1}{2}k\left(-\frac{m}{M}(x_1 + x_3) - x_1\right)^2 - \frac{1}{2}k\left(x_3 + \frac{m}{M}(x_1 + x_3)\right)^2$$

etc.

(*)
Guessing the modes ⟺ symmetry:

$1/\mu = \frac{1}{m} + \frac{1}{m} + \frac{1}{M}$

$\omega^2 = 0$; translation mode

$\omega^2 = k/m$; symmetric mode ⟹ $\eta = x_1 - x_3$

$\omega^2 = k\mu/mM$; anti-symmetric mode ⟹ $\eta = x_1 + x_3$

→ Aside : Example from Continuum.

Consider:

$$L = \int_{x_1}^{x_2} dx \, \mathcal{L}$$

$$\mathcal{L} = \frac{(\partial_t f)^2}{2} - \frac{(\partial_x f)^2}{2} - U(f)$$

i.e. $U = 0 \rightarrow$ wave equation

$U = f^2/2 \rightarrow$ Klein-Gordon

etc.

$U = \alpha f^2/2 + \frac{\beta f^4}{4} \rightarrow \phi^4$ model 1D.

(can relate magnetism).

as for NL string:

$$\text{LEM} \Rightarrow \quad \partial_t^2 f - \partial_x^2 f + \frac{\partial U}{\partial f} = 0$$

For static, eqbrm solution:

$$\partial_t^2 f_0 = 0$$

$$\Rightarrow \quad \partial_x^2 f_0 - \frac{\partial U}{\partial f_0} = 0$$

Now, for fluctn. about:

$$f = f_0 + \tilde{f}$$

$$\tilde{f} = \hat{f} e^{-i\omega t} \qquad \hat{f} = \hat{f}(x)$$

$\Rightarrow$ plug into EOM and linearize:

$$-\omega^2 \hat{f} - \partial_x^2 \hat{f} + \frac{\partial U}{\partial f}(f_0 + \hat{f}) = 0$$

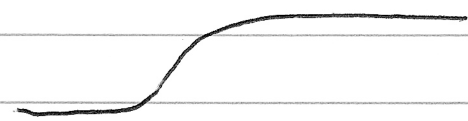
$$\Rightarrow -\omega^2 \hat{f} - \partial_x^2 \hat{f} + \frac{\partial^2 U}{\partial f^2} \hat{f} = 0$$

i.e. $\quad -\partial_x^2 \hat{f} + \left(\frac{\partial^2 U}{\partial f_0^2}\right) \hat{f} = \omega^2 \hat{f} \qquad$ eigenmode eqn.

note $\omega^2 = 0 \quad \Rightarrow \quad -\partial_x^2 \hat{f} + \left(\frac{\partial^2 U}{\partial f_0^2}\right) \hat{f} = 0$

but can also observe:

$$-\partial_x^3 f_0 + \frac{\partial U}{\partial f_0} = 0$$



$\Rightarrow$ a solution $f_0(x)$.

Now can translate that solution arbitrarily, as have translation symmetry in x ✓

i.e. $f_0(x) \to f_0(x + \delta x_0)$ — must be solution

δx_0: infinitesimal centroid shift

$$-\partial_x^3 \left(f_0(x + \delta x_0)\right) + \frac{\partial U}{\partial f}\left(f_0(x + \delta x_0)\right) = 0$$

expand in δx_0:

$$\delta x_0 \, - \partial_x^3 f_0 + \delta x_0 \, \frac{\partial^2 U}{\partial f^2}(\partial_x f_0) = 0$$

i.e. $$\frac{d}{dx}\left(-\partial_x^3 f_0 + \frac{\partial U}{\partial f_0}\right) = 0.$$

so

$$-\partial_x^2(\partial_x f_0) + \left.\frac{\partial^2 V}{\partial f^2}\right|_{f_0} (\partial_x f_0) = 0$$

$\Leftrightarrow$ but eigenmode eqn is:

$$-\partial_x^2 \hat{f} + \left.\frac{\partial^2 V}{\partial f^2}\right|_{f_0} \hat{f} = \omega^2 \hat{f}$$

so

$\omega^2 = 0$ is eigenmode with eigenfunction $\partial_x f_0$

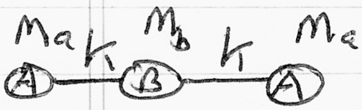
$\rightarrow$ translation mode, due translation symmetry of $\mathcal{L}$.

$\rightarrow$ obviously generalizable.

iii.) Triatomic Molecule → 2D

$3 \times 2 = 6$ modes.

a) Linear



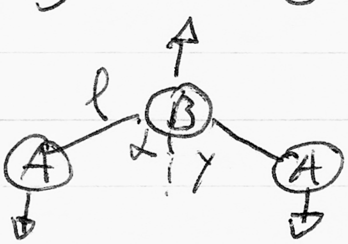
{ harmonic binding
1D → as previous example.

modes : 1) $\omega^2 = 0$ → translation $\vec{x}, y$
3 symm. → rotation with m_B fixed

{ i.e. $\underline{3}$ invariant transformations
⇒ $\underline{\underline{3}}$ zero frequency modes

(vibration) 2) linear; symmetric $\omega^2 = k/m$; $(x_1 - x_3)$
antisymmetric $(x_1 + x_3)$; $kM/m_A m_B$

(rotation) 3) bending → symmetric in x



(antisymmetric → rotation)
→ anti-symm in y

Proceeding as before:

$$m_A y_1 + m_B y_2 + m_A y_3 = 0$$

$$T = \frac{1}{2} m_A (\dot{y}_1^2 + \dot{y}_3^2) + \frac{m_B}{2} \dot{y}_2^2$$

↳ can eliminate in terms y_1, y_3

bend $\measuredangle$ of molecule

$$U = \tfrac{1}{2} k (\delta L)^2 \; ; \qquad \delta L = l_1 \cos\alpha_1 + l_2 \cos\alpha_2$$

$l_1 = l_2$

small osc

$$= l\left[\frac{(y_1 - y_2)}{l} + \frac{(y_3 - y_2)}{l}\right]$$

$$\Rightarrow \quad L = \tfrac{1}{2} m_A (\dot{y}_1^2 + \dot{y}_3^2) + \tfrac{1}{2} m_B \dot{y}_2^2 - \tfrac{1}{2} k \left[(y_1 - y_2) + (y_3 - y_2)\right]^2$$

subst for y_2

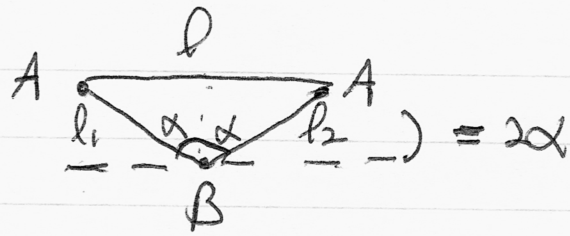
$$= \frac{m_A m_B}{4M} l^2 \dot{\delta}^2 - \tfrac{1}{2} k l^2 \delta^2$$

$$\delta = (y_1 + y_3 - 2y_2)/l$$

so

$$\boxed{\omega^2 = 2kM / m_A m_B.}$$

D.) Triangular (2D)



$\ell_1 = \ell_2$

Now → $3 \times (2) = 6$ degs. freedom

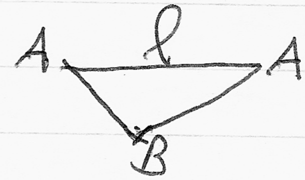
(2) → x, y

⇒ expect $\underline{\underline{6}}$ modes

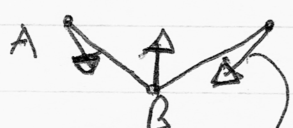
→ can immediately identify $\underline{\underline{3}}$ zero frequency modes →
- $\hat{x}$ translation
- $\hat{y}$ translation
- (centroid) rotation

[in general, each symmetry ⟺ 1 zero frequency mode]

→ can classify remaining modes by symmetry

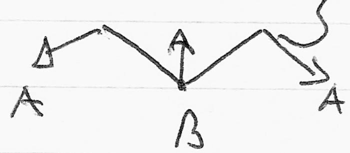


a) $\hat{y}$ symmetric modes



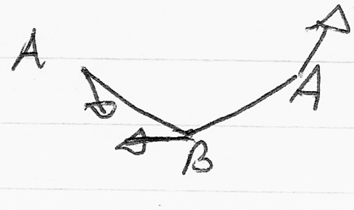
B – up
A → down, in → analogous 1D breather (+ bend)

must be orthogonal



B – up
A → down, out ⇒ analogous 1D breather (+ bend)

b) $\hat{y}$ non-symmetric mode



→ 1d asymmetric

→ To calculate;

$\underline{x} = (x, y)$

$$L = \frac{1}{2} m_A (\dot{\underline{x}}_1^2 + \dot{\underline{x}}_3^2) + \frac{1}{2} m_B \dot{\underline{x}}_2^2$$

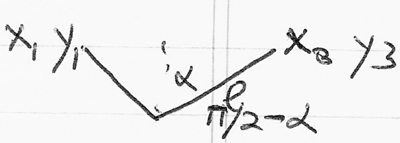
$$- \frac{1}{2} k_1 (\delta l_1^2 + \delta l_2^2) - \frac{1}{2} k_2 l^2 \delta^2$$

asymmetric
mode → need
not be equal

3 constraints: → CM stationary (2 components)
& L cons.

$$m_A(x_1 + x_3) + m_B \dot{x}_2 = 0$$
$$m_A(y_1 + y_3) + m_B y_2 = 0$$

→ $\underline{L}$ conserved



taking $\underline{L}$ about vertex (origin);

$$\underline{r}_1 = \left(-l\cos(\pi/2-\alpha),\ l\sin(\pi/2-\alpha)\right)$$

$$\underline{r}_3 = \left(l\cos(\pi/2-\alpha),\ l\sin(\pi/2-\alpha)\right)$$

$$\underline{L} = \sum_\alpha m_\alpha \underline{r}_\alpha \times \underline{v}_\alpha \Rightarrow \delta \underline{L} = 0 \Rightarrow$$

$$\sum_\alpha m_\alpha \underline{r}_\alpha \times \delta \underline{x}_\alpha = 0$$

displacement

$$\Rightarrow \ell\left[(y_1-y_3)\sin\alpha - (x_1+x_3)\cos\alpha\right] = 0$$

so 3rd constr $\Rightarrow$ $\boxed{(y_1-y_3)\sin\alpha - (x_1+x_3)\cos\alpha = 0}$

and crank.