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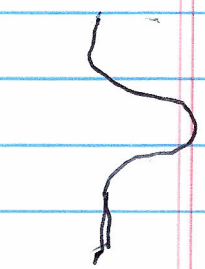
Application → Self-Focusing

Consider a beam propagating in a medium with  $n = n_0 + \Delta n |E|^2$   
Kerr liquid

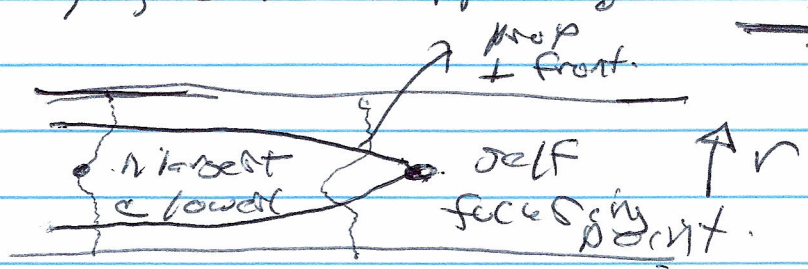
Theme: insight from mechanics

↓ intensity dependent refractive index → nonlinear

→ qualitatively, what happens?



↓ beam profile (initial state)



medium → high intensity ⇒ high n ⇒ low speed.

⇒ beam self focuses

How describe?

$$\nabla^2 E + n^2 \frac{\omega^2}{c^2} E = 0$$

$$k^2 = \frac{n^2 \omega^2}{c^2}$$

$$E = e^{ikz} \psi(z, t)$$

→

$$\begin{aligned}
 & -k_z^2 \cancel{\psi} + 2ck_z \frac{\partial \psi}{\partial z} + \cancel{h \cdot \nabla \cdot t} + \nabla_{\perp}^2 \psi \\
 & + \cancel{\Lambda_0^2 \frac{\omega^2}{c_0^2}} \psi + \Lambda_2 \frac{|\psi|^2}{c_0^2} \omega^2 \psi = 0
 \end{aligned}$$

⇒ so

$$2ck_z \frac{\partial \psi}{\partial z} + \nabla_{\perp}^2 \psi + \Lambda_2 \frac{\omega^2}{c_0^2} |\psi|^2 \psi = 0$$

- parabolic wave eqn, but nonlinear
- NLS (i.e. Hartree-self consistent field)

Now, proceed as in HW.

$$\left\{ \begin{aligned}
 & 2ck_z \frac{\partial \psi}{\partial z} + \nabla_{\perp}^2 \psi = -k^2 \frac{\Delta n}{n_0} \psi \\
 & \Delta n / n_0 = n_0 |\psi|^2 / n_0
 \end{aligned} \right.$$

$$\psi = A e^{i\phi} \quad \text{, also eikonal}$$

⇒

$$2ckz \frac{\partial \psi}{\partial z} = 2ckz \left( \frac{\partial A}{\partial z} e^{i\phi} + A i \frac{\partial \phi}{\partial z} \right)$$

$$\begin{aligned} \nabla_{\perp}^2 \psi &= \nabla_{\perp}^2 (A e^{i\phi}) = \nabla_{\perp} (A i \nabla_{\perp} \phi e^{i\phi} + \nabla_{\perp} A e^{i\phi}) \\ &= (A i \nabla_{\perp}^2 \phi + i 2 \nabla_{\perp} \phi \cdot \nabla_{\perp} A e^{i\phi} - A (\nabla_{\perp} \phi)^2 \\ &\quad + \nabla_{\perp}^2 A) e^{i\phi} \end{aligned}$$

~~Real~~  $\text{Real} \rightarrow \text{phase}$   
 $\text{cm} \rightarrow \text{amp.}$

$$\begin{aligned} + (2ckz) (i) A \frac{\partial \phi}{\partial z} e^{i\phi} - A (\nabla_{\perp} \phi)^2 e^{i\phi} \\ + \nabla_{\perp}^2 A e^{i\phi} + k^2 \frac{\Delta n}{n_0} A e^{i\phi} \end{aligned}$$

⇒

$$\frac{\partial \phi}{\partial z} + \frac{1}{2k} (\nabla_{\perp} \phi)^2 - \frac{k}{2} \left( \frac{\nabla_{\perp}^2 A}{k^2 A} + 2 \frac{\Delta n}{n} \right) = 0$$

811

$$\frac{\partial \phi}{\partial z} + \frac{1}{2k} (\nabla_{\perp} \phi)^2 - \frac{k}{2} \left( \frac{\nabla_{\perp}^2 A}{k^2 A} + \frac{2\Delta n}{n} \right) = 0$$

phase

$$k \frac{\partial}{\partial z} A^2 = -\nabla_{\perp} \cdot (A^2 \nabla_{\perp} \phi)$$

amplitudeamplitude

Now, given paraxial wave eqn  $\leftrightarrow$  Schrodinger eqn.

no surprise that phase equation resembles H-J eqn.

i.e.

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial \mathbf{r}} \right)^2 + V = 0$$

thus, can make analogy with particle dynamics, i.e.

$$V = -\frac{k}{2} \left( \frac{\nabla_{\perp}^2 A}{k^2 A} + \frac{2\Delta n}{n} \right)$$

For dynamics:

$$z \leftrightarrow t$$

$$r \leftrightarrow z$$

So

$$m \frac{d^2 z}{dt^2} = - \frac{\partial V}{\partial z}$$

⇒

$$\frac{d^2 r}{dz^2} = - \frac{1}{k} \frac{\partial V}{\partial r}$$

ray path  
of radius  $r$   
vs  $z$   
( $r$ -side)

1<sup>st</sup>

$$\frac{d^2 r}{dz^2} = + \frac{\partial}{\partial r} \left[ \frac{1}{2} \left( \frac{\sigma_{\perp}^2 A}{k^2 A} + 2 \frac{\Delta n}{n} \right) \right]$$

dynamical problem.

Now - formally particle dynamics, in the UoA.

- but need  $A$

-  $A$  from coupled eqns.  $\phi, A$ ; involving  $\Delta n$ .

To make progress try approximate  
form for  $A(r, z)$ :

d.e

$$A(r, z) = A_0 \frac{P_0^2}{a^2(z)} \exp\left[\frac{-r^2}{2a^2(z)}\right]$$

- paraxial approx

- each ray follows  $r/a \sim r_0/a_0$   
d.c.

so plugging in:

$$V = -k \left( \frac{-1}{k^2 a^2} + \frac{r^2}{2k^2 a^4} + \frac{\Delta n}{n_0} \right)$$

Now, for paraxial  $\rightarrow$  most beam on axis

$$\text{so } r^2 \ll a^2$$

$$\Rightarrow V(a) = -k \left( \frac{-1}{k^2 a^2} + \frac{\Delta n}{n_0} \right)$$

So, 
$$V(a) = -k \left( \frac{-1}{k^2 a^2} + \frac{\Delta n}{n_0} \right)$$

and

$$\frac{d^2 r}{dz^2} = -k^{-1} \frac{\partial V}{\partial r}$$

becomes

$$k \frac{d^2 a(z)}{dz^2} = - \frac{\partial V}{\partial a}$$

\*  $a' = da/dz$

$$k \frac{da}{dz} \frac{d^2 a}{dz^2} + \frac{da}{dz} \frac{\partial V}{\partial a}$$

$$\frac{d}{dz} \left\{ \frac{k}{2} \left( \frac{da}{dz} \right)^2 + V(a) \right\} = 0$$

so at last:

$$\boxed{\frac{k}{2} \left( \frac{da}{dz} \right)^2 + V(a) = \text{const.}}$$

↓  
K.E.

↓  
potential

energy  
conservation  
relation

def-focal  $\Rightarrow V < 0$   
 $\Rightarrow$  pulls in ray

$$\frac{k}{2} \left( \frac{dq}{dz} \right)^2 + V(a_0) = \frac{k}{2} \left( \frac{dq}{dz} \right)^2 + V(a)$$

⇒

$$Z_f = \int_{a_0}^{a_{min}} \left\{ \frac{2}{k} (V(a_0) - V(a)) + \left( \frac{dq}{dz} \right)^2 \right\}^{-1/2} dz$$

↓  
focal length

Key competition:

$$V = -k \left[ -\frac{1}{k^2 a^2} + n_2 \frac{A_0^2}{n_0} \right]$$

↓  
diffraction  
(repulsive  
pot.)

↓  
(attr.  
potential)

∞

$$V = \frac{7}{k a^2} \left( 1 - \frac{P}{P_0} \right)$$

self-focusing  
for  
 $P > P_0$  !

$$P = n_0 c a^2 A_0^2 / 2 \rightarrow \text{beam power}$$

$$P_0 = c \lambda^2 / 8 \pi n_2$$



crank

⇒

$$Z_f = \frac{k a_0^2 / \sqrt{2}}{\left( \frac{P}{P_0} - 1 \right)^{1/2} - \left( \frac{k a_0}{\sqrt{2}} \right) \left( \frac{da}{dz} \right)}$$

focal length.

Now, for exact result:  $\left\{ \begin{array}{l} L-L / \text{Section 109} \\ \text{Vlasov, Petrishov,} \\ \text{Tolstov, 1976} \end{array} \right.$

$$i\hbar \frac{\partial}{\partial z} \psi + \frac{1}{2} \nabla_{\perp}^2 \psi + \eta |\psi|^2 \psi = 0$$

⇒ prob. cons. analogue of prob. current

$$\frac{\partial}{\partial z} |\psi|^2 + \nabla \cdot \underline{J} = 0$$

$$\underline{J} = \frac{c}{2k_0} \left[ \psi \nabla_{\perp} \psi^* - \psi^* \nabla_{\perp} \psi \right]$$

Can show 2 conserved quantities:

$$N = \int d^3r |\psi|^2 \rightarrow \#, \text{ probability}$$

$$\varepsilon = \frac{1}{2k^2} \int d^2r \left\{ |\sigma_{\perp} \psi|^2 - \eta (|\psi|^2)^2 \right\}$$

↓
(direct)

energy

Now, can define mean beam radius:

$$R^2(z) = \frac{1}{N} \int d^2r r^2 |\psi|^2 \quad \langle r^2 \rangle$$

and  $\frac{d}{dz} R^2(z) \Rightarrow$

$$R^2(z) = 2\varepsilon (z - z_0)^2 / N + R_0^2$$

$\Rightarrow \varepsilon < 0 \Rightarrow$  collapse to focal point