General Continuum Dynamics

→ An Introduction

f.) More Oscillations : Mechanics of Fields

→ recall the string : (i.e. continuum limit)

$\mathcal{L} = \mathcal{L}(y, y_t, y_x)$ → Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu y_t^2 - T\Big[\underbrace{(1+y_x^2)^{1/2} - 1}_{\text{potential energy in string}}\Big] \qquad \underline{\underline{(1D!)}}$$

where $S = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$ t, x both parameters

Then, for EOM : $\delta S = 0$ (as usual)

$$\delta S = 0 = \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\delta y_x \right)$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\frac{d}{dt}\delta y + \frac{\partial \mathcal{L}}{\partial y_x}\frac{d}{dx}\delta y \right)$$

$$= \underbrace{\cancel{\int_0^L dx \frac{\partial \mathcal{L}}{\partial y_t}\delta y \Big|_{t_1}^{t_2}}}_{\to 0} + \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial y_x}\delta y \Big|_0^L$$

fixed end pts in time!

$$+ \int_{t_1}^{t_2}\int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right)\delta y$$

thus, have Lagrange EOM:

$$\boxed{\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial y_t}\right) = \dfrac{\partial \mathcal{L}}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{L}}{\partial y_x}\right) \\[2ex] \text{with B.C.:} \qquad \left.\dfrac{\partial \mathcal{L}}{\partial y_x}\,\delta y\right|_0^L = 0 \end{array}}$$

(clear for fixed, free ends)

n.b.: → have

- spatial ibp endpt. $\displaystyle\int_{t_1}^{t_2} dt \left.\frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\right|_0^L$

- $\delta y(t, x) = 0$, all x, only at t_2, t_1.

→ in 3D, have:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial x_i}\right)$$

→ for 1D string:

$$\frac{d}{dt}\left(\mu\, y_t\right) = \frac{d}{dx}\left(\frac{T\, y_x}{(1+y_x^2)^{1/2}}\right)$$

small oscillations : $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{T}{2}(y_x)^2$

$\therefore$

$\mu y_{t,t} = T y_{x,x}$ $\Rightarrow$ garden variety wave eqn.

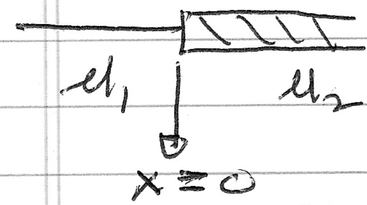
$\rightarrow$ Ex. $U(\phi) = \alpha\frac{\phi^2}{2} + \beta\phi^4$

$$\mathcal{L} = \frac{\phi_t^2}{2} - \frac{(\nabla\phi)^2}{2} - U(\phi)$$

$\Rightarrow$ EOM ?

Now, Lagrangian formulation allows unambiguous formulation of basic equations for matching;

$\Rightarrow$ consider $\underline{\underline{2}}$ prototypical examples i.)



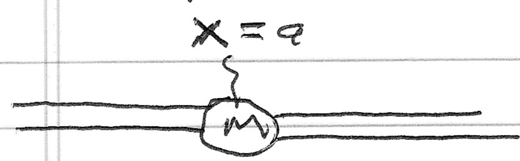
matching $\Rightarrow$ $y_-(0) = y_+(0)$

$$\int_{0_-}^{0_+} \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{\partial \mathcal{L}}{\partial y} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right\} = 0$$

i.e. integrate EOM

$$\left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_+} = \left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_-}$$

ii.)



(continuity understood)

$\mu \rightarrow \mu + M\delta(x-a)$

$$\mathcal{L} = \frac{1}{2}\left(\mu + M\delta(x-a)\right) y_t^2 - \frac{T}{2} y_x^2$$

$$(\mu + M\delta(x-a))\, y_{tt} = T y_{xx}$$

$$y = \tilde{y}(x)\, e^{-i\omega t}$$

$$T\hat{y}_{xx} = -\omega^2(\mu + M\delta(x-a))\hat{y}$$

$$\int_{a_-}^{a_+}\left[T\hat{y}_{xx} + \omega^2(\mu + M\delta(x-a))\hat{y}\right] = 0$$

$$T\,\hat{y}_x \Big|_{a_-}^{a_+} = -\omega^2 M\,\hat{y}(a) \qquad \Rightarrow \text{jump condition.}$$

N.B.: Use of Lagrangian ab-initio renders all questions re: order of derivatives moot.

→ Hamiltonian Fomulation

As usual, can define canonical momentum

$$\boxed{\pi = \partial \mathcal{L} / \partial y_t} \quad \Rightarrow \mu y_t \quad \Rightarrow \begin{array}{c}\text{momentum}\\ \text{of string element}\end{array}$$

N.B. $\pi = \mu y_t$, for string. $(y_t = \dot{y})$

Similarly, can define Hamiltonian density:

$$\mathcal{H} = \pi y_t - \mathcal{L} = \pi y_t - \mathcal{L} \quad \text{for } \partial\mathcal{L}/\partial t = 0, \quad \underbrace{\mathcal{H} = \mathcal{E}}_{\text{energy density}}$$

and Hamiltonian $H = \int_0^L dx\, \mathcal{H}$.

For string:

$$\mathcal{H} = \frac{\pi^2}{\mu} - \mathcal{L} = \underbrace{\frac{\pi^2}{2\mu}}_{KE} + \underbrace{\frac{T(y_x)^2}{2}}_{\text{Pot. E.}}$$

Hamilton's Eqns then follow from Principle of Least Action, i.e.

$$S = \int_{t_1}^{t_2} dt \int_0^L dx \, (\pi y_t - \mathcal{H}) \qquad \left\{ \begin{array}{l} \mathcal{L} = \pi y_t - \mathcal{H} \\ \mathcal{H} = \mathcal{H}(\pi, y_x, y) \\ \mathcal{L} = \mathcal{L}(y_t, y_x, y) \end{array} \right.$$

so

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left(y_t \, \delta\pi + \pi \, \delta y_t - \left(\frac{\partial \mathcal{H}}{\partial \pi} \delta\pi + \frac{\partial \mathcal{H}}{\partial y_x} \delta y_x + \frac{\partial \mathcal{H}}{\partial y} \delta y \right) \right)$$

ignoring surface terms

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ y_t \, \delta\pi - \left(\frac{d\pi}{dt} \right) \delta y - \cancel{\frac{\partial \mathcal{H}}{\partial y}} - \frac{\partial \mathcal{H}}{\partial \pi} \delta\pi - \left(\frac{\partial \mathcal{H}}{\partial y} - \frac{d}{dx} \left(\frac{\partial \mathcal{H}}{\partial y_x} \right) \right) \delta y \right\}$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \left(\dot{y} - \frac{\partial \mathcal{H}}{\partial \pi} \right) \delta\pi + \delta y \left(\frac{\partial \mathcal{H}}{\partial y} - \frac{d}{dx} \left(\frac{\partial \mathcal{H}}{\partial y_x} \right) + \frac{d\pi}{dt} \right) \right\}$$

$\delta S = 0 \Rightarrow$

$$\begin{cases} \dot{y} = \dfrac{\partial \mathcal{H}}{\partial \pi} \\[2ex] \dot{\pi} = -\dfrac{\partial \mathcal{H}}{\partial y} + \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{H}}{\partial y_x}\right) \end{cases}$$

and can observe further; $\left(\partial_t \mathcal{L} = 0 \text{ here}\right)$

$$\mathcal{H} = \pi y_t - \mathcal{L}$$

$$\frac{d\mathcal{H}}{dt} = \pi \ddot{y} + \dot{\pi} \dot{y} - \frac{d\mathcal{L}}{dt}$$

$$= \pi \ddot{y} + \dot{\pi} \dot{y} - \left(\frac{\partial \mathcal{L}}{\partial y} \dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{y}} \ddot{y} + \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x \right)$$

but $\pi = \dfrac{\partial \mathcal{L}}{\partial y_t}$

$$\Rightarrow \frac{d\mathcal{H}}{dt} = \dot{\pi} \dot{y} - \left(\frac{\partial \mathcal{L}}{\partial y} \dot{y} + \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x \right)$$

further, from LEOM:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

$\Rightarrow$

$$\frac{d\mathcal{H}}{dt} = \dot{\pi}\dot{y} - \dot{y}\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)\right) - \frac{\partial \mathcal{L}}{\partial y_x}\dot{y}_x$$

since $\pi = \partial \mathcal{L}/\partial \dot{y}$

$\Rightarrow$

$$\frac{d\mathcal{H}}{dt} = -\dot{y}\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial \mathcal{L}}{\partial y_x}\frac{\partial}{\partial x}\dot{y}$$

$$= -\frac{d}{dx}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

thus have shown, in general:

$$\boxed{\frac{d\mathcal{H}}{dt} + \frac{d}{dx}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0}$$

and in higher dimensions:

$$\boxed{\frac{d\mathcal{H}}{dt} + \sum_i \frac{\partial}{\partial x_i}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right) = 0}$$

Thus, have shown (in general) $\Rightarrow$

$$\frac{d\mathcal{H}}{dt} + \frac{\partial}{\partial x}\left(\dot{y}\,\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0$$

and can generalize to higher dimensions

$$\boxed{\frac{d\mathcal{H}}{dt} + \sum_i \frac{\partial}{\partial x_i}\left(\dot{y}\,\frac{\partial \mathcal{L}}{\partial y_{x_i}}\right) = 0}$$

$(*)$

What does it mean?

Here $\mathcal{H} = \mathcal{E} \equiv$ energy density

so relation of form:

$$\frac{d\mathcal{H}}{dt} + \underline{\nabla} \cdot \underline{S} = 0 \quad ; \quad S_i = \dot{y}\,\frac{\partial \mathcal{L}}{\partial y_{x_i}}$$

Poynting theorem!, with:
(Energy)

$$S_x = \dot{y}\,\frac{\partial \mathcal{L}}{\partial y_x}$$

as $\left\{\begin{array}{l}\text{Poynting flux} \\ \text{i.e. wave energy density flux in} \\ \quad \text{direction of wave propagation}\end{array}\right.$

For string

$$S_x = \dot{y}\,\partial \mathcal{L}/\partial y_x = -T\,\dot{y}\,y_x$$

Note:

→ Poynting thm. relates (local) wave energy density with wave energy density flux, i.e.

$$\frac{d\mathcal{H}}{dt} + \partial_x S_x = 0$$

→ Poynting thm. relates rate of energy change to wave energy density flux thru interval

i.e.

$$\frac{d}{dt} E = \frac{d}{dt} \int_{x_1}^{x_2} \mathcal{H} \, dx = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} S_x$$

$$= -S_x \Big|_{x_1}^{x_2}$$

→ Poynting thm. formed by expressing $\frac{d}{dt} \mathcal{E}$ as $\underline{\nabla} \cdot \underline{S}$, etc.

recall in E and M:

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c} \, \partial \underline{B} / \partial t$$

but $\mathcal{E} = E^2/8\pi + B^2/8\pi$

then $\left(\frac{\partial \underline{E}}{\partial t} = c \underline{\nabla} \times \underline{B} - 4\pi \underline{J} \right) \cdot \underline{E}/4\pi$

$\left(\frac{\partial \underline{B}}{\partial t} = - c \underline{\nabla} \times \underline{E} \right) \cdot (\underline{B}/4\pi)$

$\Rightarrow$

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) = - \underline{E} \cdot \underline{J} - \underline{\nabla} \cdot \left(\underbrace{\frac{c}{4\pi} \underline{E} \times \underline{B}}_{\underline{S}} \right)$$

i.e. form Poynting thm. by considering time rate of change of energy density.

$\rightarrow$ Important to distinguish:

(particle)

$\Pi = \mu \dot{y} \hat{y} \equiv$ canonical momentum

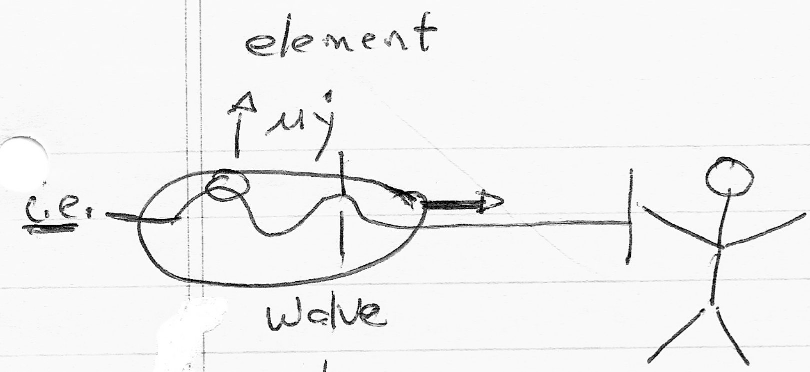
$\rightarrow$ momentum of string element $\mu \dot{y}(x,t)$, in $\hat{y}$ direction

(quasi-particle)

$$\underline{S} = - T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \hat{x} = \frac{\partial \mathcal{L}}{\partial y_x} \frac{\partial y}{\partial t} \hat{x}$$

$\equiv$ wave energy density flux

$\rightarrow$ momentum of wave / fluctuation, in $\hat{x}$ direction



$\rightarrow$ "feels" wave momentum (kick in $\vec{x}$)

flux thru (x) at time t $\equiv S_x(x,t)$

calculating for wave on string:

of $y = A\cos(k(x - v_{ph}t))$ $\qquad v_{ph} = (T/\mu)^{1/2}$

$$\frac{\partial y}{\partial t} = +A k v_{ph} \sin(k(x - v_{ph}t))$$

$$\frac{\partial y}{\partial x} = -Ak\sin(k(x - v_{ph}t))$$

$$S_x = +T A^2 k^2 v_{ph} \sin^2(k(x - v_{ph}t))$$

$$\therefore \quad \overline{S_x} = \frac{Tk^2}{2} v_{ph} A^2$$

but: $\omega^2 = v_{ph}^2 k^2$

$$\boxed{\overline{S_x} = \frac{\mu\omega^2}{2} v_{ph} A^2}$$

$\overline{S_x} = v_{gr}\,\mathcal{E}$

as $v_{gr} = v_{ph}$

$\mathcal{E} = 2 * \overline{KE}$

$= 2 * \frac{1}{4}\mu\omega^2 A^2$

→ Wave Momentum

- have developed notions of wave energy and Poynting Theorem, ∴

- natural to investigate wave momentum.

Now, recall in E+M,

$$\underline{P}_{EM} = \frac{1}{c^2}\underline{S} = \frac{1}{4\pi c}\underline{E}\times\underline{B}$$

$\underline{P}_{EM}$ ↓ momentum of electromagnetic wave

$\underline{S}$ ↳ Poynting vector

Thus, natural motivation to investigate relation for string, i.e.

$$S = \dot{y}\frac{\partial \mathcal{L}}{\partial y_x}$$

so

$$\dot{S} = \ddot{y}\frac{\partial \mathcal{L}}{\partial y_x} + \dot{y}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

for string;

$$\ddot{y} = \frac{T}{\mu}y_{xx} = V_{ph}^2 y_{xx} \quad ; \quad \frac{\partial \mathcal{L}}{\partial y_x} = -T y_x$$

$$\dot{S} = \left\{ -\frac{T}{\mu} y_{xx} T y_x - \mu \dot{y} \frac{T}{\mu} \frac{d}{dt} y_x \right\}$$

$$= -\frac{T}{\mu} \frac{\partial}{\partial x} \left\{ \frac{T y_x^2}{2} + \frac{\mu \dot{y}^2}{2} \right\}$$

$$= -V_{ph}^2 \frac{\partial}{\partial x} \mathcal{E}$$

$$\mathcal{E} = \omega \frac{\mathcal{E}}{\omega} = \omega N$$

$$P_w = \frac{1}{V_{ph}^2} S = \frac{k}{\omega} \mathcal{E} = kN$$

semiclassical analogy

$\therefore$ if define wave momentum density $\underline{P}_w = \frac{1}{V_{ph}^2} \underline{S}$,

have natural conservation law

$$\boxed{\frac{d}{dt} \underline{P}_w + \underline{\nabla} \mathcal{H} = 0}$$

{ akin ponderomotive force

here $\underline{\nabla} \mathcal{H} = \underline{\nabla} \mathcal{E}$ is force density. Then

wave stress

(pushes in direction of propagation)

for $\underline{P} = \int_{x_1}^{x_2} dx \, \underline{P}_w$

wave momentum in a chunk of string,

$$\frac{d}{dt} P + \mathcal{H} \Big|_{x_1}^{x_2} = 0$$

difference/jump in energy density across chunk.

Thus, have complete energy, momentum relations.

$$\boxed{\begin{aligned} &\frac{d}{dt}\mathcal{H} + \underline{\nabla}\cdot\underline{S} = 0 && \underline{S} = \cdot\, \dot{y}\,\frac{\partial\mathcal{L}}{\partial y_x}\,\hat{i}_w \\ &\frac{\partial}{\partial t}\underline{P}_w + \underline{\nabla}\mathcal{H} = 0 && \underline{P}_w = \frac{1}{v_{ph}^2}\,\underline{S}\,\hat{i}_w \end{aligned}}$$

→ Can derive from divergence relation for stress tensor (E&M).

An application: Sound

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{V}) = 0 \qquad P = P(\rho)$$

$$\frac{dP}{d\rho} \equiv c_s^2$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{1}{\rho}\,\underline{\nabla}P$$

linearizing ⇒

$$\frac{\partial\hat{\underline{V}}}{\partial t} = \frac{-1}{\rho_0}\,c_s^2\,\underline{\nabla}\hat{\rho}$$

$$\frac{\partial\hat{\rho}}{\partial t} = -\rho_0\,\underline{\nabla}\cdot\hat{\underline{V}}$$

Note: Can write:

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot S = 0$$

$$\frac{\partial}{\partial t} P_w + \nabla \mathcal{H} = 0$$

transpose

$$\text{so} \quad \begin{pmatrix} \frac{1}{v_{ph}}\partial/\partial t \\ \partial/\partial x \end{pmatrix}^{T} \begin{bmatrix} \mathcal{H} & S/v_{ph} \\ S/v_{ph} & \mathcal{E} \end{bmatrix} \equiv 0$$

$\mathcal{E} = \mathcal{H}$, here.

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu} \equiv$ energy-momentum tensor of string

$$\partial_\mu = \left(\frac{1}{v_{ph}}\partial_t,\ \partial_x\right)$$

For E+M:

$$T^{ik} = \begin{pmatrix} W & S_x/c & S_y/c & S_z/c \\ S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left\{ -E_\alpha E_\beta - H_\alpha H_\beta + \frac{1}{2}\delta_{\alpha\beta}(E^2+H^2) \right\}$$

$\downarrow$

Maxwell stress tensor.

Then: $\frac{\partial^2 \hat{\rho}}{\partial t^2} = C_s^2 \nabla^2 \hat{\rho} = \rho_0 \nabla \cdot \left\{ \frac{C_s^2}{\rho_0} \nabla \rho \right\}$

For energy-momentum relations:

$(1) \cdot \hat{V}\rho_0 + (2) \cdot \frac{\hat{\rho} C_s^2}{\rho_0} \Rightarrow$ $\hat{V} \cdot \nabla \hat{\rho}$

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2}\right) + C_s^2 \qquad = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\hat{\rho}^2 C_s^2}{2\rho_0}\right) + C_s^2 \hat{\rho}\, \nabla \cdot V = 0$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2} + \frac{\hat{\rho}^2 C_s^2}{2\rho_0}\right) + \nabla \cdot \left[C_s^2 \rho V \right] = 0$$

$$H = \mathcal{E} = \underbrace{\frac{\rho_0 \hat{V}^2}{2}}_{T} + \underbrace{\frac{\hat{\rho}^2 C_s^2}{2\rho_0}}_{V} \qquad \underline{S}$$

Similarly,

$$\underline{P}_w = \frac{1}{C_s^2} \underline{S}$$

$$\frac{\partial P_w}{\partial t} = \frac{\partial}{\partial t}(\rho V) = \frac{\partial \hat{\rho}}{\partial t} \hat{V} + \hat{\rho} \frac{\partial \hat{V}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \hat{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \hat{\underline{V}}$$

$$\frac{\partial \hat{\underline{V}}}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \hat{\rho}$$

$$\therefore$$

$$\frac{\partial P_w}{\partial t} = -\frac{\rho_0}{2} \underline{\nabla} \cdot [\underline{V}\,\underline{V}] - \frac{c_s^3}{\rho_0} \underline{\nabla} \cdot \left(\frac{\hat{\rho}^3}{2} \right)$$

$$= -\underline{\nabla} \left(\frac{\rho V^2}{2} + \frac{c_s^2}{\rho_0} \frac{\hat{\rho}^3}{2} \right) \quad \checkmark$$

for longitudinal waves.