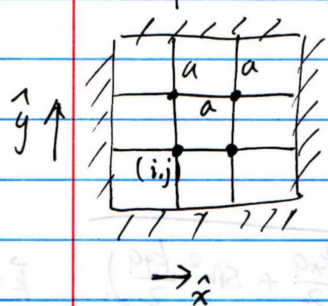
Yinming Shao

1. (FW 4.16)



(a) Let η_{ij} denote the small transverse displacements of the masses.

$$KE = \frac{1}{2} m \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{\eta}_{ij}^2$$

$$PE = \frac{1}{2} k \left[(\delta^x \eta)^2 + (\delta^y \eta)^2 \right] \qquad k \to \frac{\tau}{a}$$

$$\cancel{= \frac{\tau}{2a} \Big[\sum_{i=1}^{N} \sum_{j=0}^{N}} \qquad \begin{cases} \delta^x \eta = \sum_{i=1}^{N} \sum_{j=0}^{N} (\eta_{ij+1} - \eta_{ij})^2 \\ \delta^y \eta = \sum_{j=1}^{N} \sum_{i=0}^{N} (\eta_{i+1j} - \eta_{ij})^2 \end{cases}$$

$$L = T - V$$
$$= \frac{1}{2} m \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{\eta}_{ij}^2 - \frac{\tau}{2a} \left[\sum_{i=1}^{N} \sum_{j=0}^{N} (\eta_{ij+1} - \eta_{ij})^2 + \sum_{j=1}^{N} \sum_{i=0}^{N} (\eta_{i+1j} - \eta_{ij})^2 \right]$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\eta}_{ij}} = m \ddot{\eta}_{ij}$$

$$\frac{\partial L}{\partial \eta_{ij}} = -\frac{\tau}{a} \left[-(\eta_{ij+1} - \eta_{ij}) + (\eta_{ij} - \eta_{ij-1}) - (\eta_{i+1j} - \eta_{ij}) + (\eta_{ij} - \eta_{i-1j}) \right)$$

$$\boxed{\text{EOM}: \quad m \ddot{\eta}_{ij} + \frac{4\tau}{a} \eta_{ij} - \frac{\tau}{a} (\eta_{ij+1} + \eta_{ij-1} + \eta_{i+1j} + \eta_{i-1j}) = 0 \quad (*)}$$

Let $x_i = ia$, $y_j = ja$

trial solution: $\eta_{ij} = \eta(x_i, y_j, t) = A e^{i(k_x x_i + k_y y_j - \omega t)}$

Plug into EOM (*):

$$-m\omega^2 \eta + \frac{4\tau}{a} \eta - \frac{\tau}{a} \left(e^{ik_x a} + e^{-ik_x a} + e^{ik_y a} + e^{-ik_y a} \right) \eta = 0$$

So: $\omega^2 = \frac{4\tau}{ma} - \frac{\tau}{ma}(2\cos k_x a + 2\cos k_y a)$

$$= \frac{2\tau}{ma}\left[2 - \cos k_x a - \cos k_y a\right]$$

$$= \frac{4\tau}{ma}\left(\sin^2\frac{k_x a}{2} + \sin^2\frac{k_y a}{2}\right)$$

Dispersion relation: $\omega(\vec{k}) = \sqrt{\frac{4\tau}{ma}\left(\sin^2\frac{k_x a}{2} + \sin^2\frac{k_y a}{2}\right)}$ $\quad \vec{k} = k_x\hat{x} + k_y\hat{y}$

(b) Continuum mass distribution.

$$\eta(x_i, y_j, t) \rightarrow u(x, y, t), \quad a \to 0$$

mass density of string $= \frac{m}{a} = \sigma = \text{const}$.

EOM: $\ddot{\eta}_{ij} = \frac{\tau}{\sigma}\left[\frac{1}{a}\left(\frac{\eta_{ij+1} - \eta_{ij}}{a} - \frac{\eta_{ij} - \eta_{ij-1}}{a}\right) + \frac{1}{a}\left(\frac{\eta_{i+1j} - \eta_{ij}}{a} - \frac{\eta_{ij} - \eta_{i-1j}}{a}\right)\right]$

$$\frac{\eta_{ij+1} - \eta_{ij}}{a} = \frac{\eta_{ij+1} - \eta_{ij}}{y_{j+1} - y_j} \xrightarrow{a \to 0} \frac{\partial u}{\partial y}$$

$$\frac{\eta_{i+1j} - \eta_{ij}}{a} = \frac{\eta_{i+1j} - \eta_{ij}}{x_{i+1} - x_i} \xrightarrow{a \to 0} \frac{\partial u}{\partial x} \qquad \ddot{\eta}_{ij} = \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} \rightarrow \text{2D - wave Eq.}$$

$$c^2 = \tau/\sigma$$

Dispersion relation:

$$\omega^2 \simeq \frac{4\tau}{ma} \cdot \frac{1}{4}\left(k_x^2 a^2 + \frac{1}{4}k_y^2 a^2\right)$$

$$= \frac{\tau a}{m}\left(k_x^2 + k_y^2\right) = c^2 k^2$$

$\boxed{\omega = c|\vec{k}|}$ travelling waves.

FW 4.16

(c) $\omega = \sqrt{\frac{\tau}{\sigma}}|\vec{k}| = c\,|\vec{k}|$ $\rightarrow$ isotropic

ω only depends on the length of the wave vector $(\vec{k} = k_x\hat{x} + k_y\hat{y})$

$\omega = \sqrt{\frac{4\tau}{ma}} \cdot \sqrt{\sin^2(\frac{k_x a}{2}) + \sin^2(\frac{k_y a}{2})}$ $\longrightarrow$ anisotropic

ω depends on direction of $\vec{k}$ in the x-y plane.

(d) Finite system, find exact normal-mode frequencies.

Let $\eta(x_i, y_j, t) = B e^{-i\omega t}[e^{ik_x x_i + ik_y y_j} - e^{-ik_x x_i + ik_y y_j} + e^{-ik_x x_i - ik_y y_j} - e^{ik_x x_i - ik_y y_j}]$

So that $\eta(0, y_j) = 0$ & $\eta(x_i, 0) = 0$ are automatically satisfied.

another two b.c. $\eta((N+1)a, y_j) = 0$ ①

$\eta(x_i, (N+1)a) = 0$ ②

① $\rightarrow$ $(e^{ik_x(N+1)a} - e^{-ik_x(N+1)a})(e^{ik_y y_j} - e^{-ik_y y_j}) = 0$

valid for any y_j

So: $\sin k_x (N+1)a = 0$

$k_x = \frac{n\pi}{(N+1)a}$ $(n = 1, 2, \cdots N)$

② $\rightarrow$ $k_y = \frac{m\pi}{(N+1)a}$ $(m = 1, 2, \cdots N)$

$$\boxed{\text{Normal mode freq. } \omega_{nm} = \sqrt{\frac{4\tau}{ma}} \cdot \sqrt{\sin^2\frac{n\pi}{2(N+1)} + \sin^2\frac{m\pi}{2(N+1)}}}$$

Continuum Limit:

$$\omega_{nm}^2 \simeq \frac{4\tau}{ma}\cdot\left(\left[\frac{n\pi}{2(N+1)}\right]^2+\left[\frac{m\pi}{2(N+1)}\right]^2\right)$$

$$= \frac{\tau}{ma}\cdot a^2\cdot\left[\left(\frac{n\pi}{l_x}\right)^2+\left(\frac{m\pi}{l_y}\right)^2\right] \qquad l_x=(N+1)a=l_y$$

$$c^2=\frac{\tau}{\sigma}$$

$$= c^2(k_x^2+k_y^2)$$

(e) Discrete Lagrangian

$$L=\frac{m}{2a^2}\sum_{i=1}^{N}\sum_{j=1}^{N}a^2\dot{\eta}_{ij}^2-\frac{\tau}{2}\left[\sum_{i=1}^{N}\sum_{j=0}^{N}a^2\left(\frac{\eta_{i,j+1}-\eta_{ij}}{a}\right)^2+\sum_{j=1}^{N}\sum_{i=0}^{N}a^2\left(\frac{\eta_{i+1,j}-\eta_{ij}}{a}\right)^2\right]$$

Continuum limit: $N\to\infty$, $a\to 0$, $\eta_{ij}\to u(x,y,t)$

$\frac{m}{a^2}\to\mu$, mass density

$$\sum_{i=1}^{N}\sum_{j=1}^{N}a^2\to\sum_i\Delta x_i\sum_j\Delta y_j=\int_0^l dx\int_0^l dy \qquad l=(N+1)a$$

So: $L=\frac{\mu}{2}\int_0^l\int_0^l u_t^2\,dx\,dy-\frac{\tau}{2}\int_0^l\int_0^l(u_x^2+u_y^2)\,dx\,dy$

$$\boxed{\text{Lagrangian density: } \mathcal{L}=\frac{\mu}{2}u_t^2-\frac{\tau}{2}u_x^2-\frac{\tau}{2}u_y^2}$$

Plug into Lagrangian eq. $\frac{\partial\mathcal{L}}{\partial u}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial u_t}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y}=0$

$\Rightarrow \quad \sigma u_{tt}-\tau u_{xx}-\tau u_{yy}=0$

$\Rightarrow \quad \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

$\boxed{4.6}$ Normally for a central potential,

Devin Cela

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$r \to r_0 + \epsilon$, $\phi \to \Omega t + \gamma$

can change of potential and ignore

$$L \approx \frac{1}{2}m(\dot{\epsilon}^2 + (r_0+\epsilon)^2(\Omega+\dot{\gamma})^2) - V(r_0) - V'(r_0)\epsilon - \frac{1}{2}V''(r_0)\epsilon^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\epsilon}} - \frac{\partial L}{\partial \epsilon} = 0 = m\ddot{\epsilon} - m(r_0+\epsilon)(\Omega+\dot{\gamma})^2 + V'(r_0) + V''(r_0)\epsilon$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} - \frac{\partial L}{\partial \gamma} = 0 = \frac{d}{dt}\Big(\underbrace{m(r_0+\epsilon)^2(\Omega+\dot{\gamma})}_{\text{constant}}\Big) = 0$$

Equations of Motion

At equilibrium, $\epsilon = \dot{\epsilon} = \ddot{\epsilon} = 0$, $\gamma = \dot{\gamma} = 0$ (no oscillations)

plugging in: $\frac{d}{dt}(m r_0^2 \Omega) = 0 \to m r_0^2 \Omega = \text{constant} \to m r^2 \dot{\phi} = \text{constant } \ell$

$-m r_0 \Omega^2 + V'(r_0) = 0 \to \left.\frac{dV}{dr}\right|_{r_0} = m r_0 \Omega^2 = m r \dot{\phi}^2$ for circular orbits

Rewriting equation of motion:

$$m\ddot{\epsilon} - \frac{\ell^2}{mr^3} + V'(r_0) + V''(r_0)\epsilon = 0$$

Expanding our second term:

$$m\ddot{\epsilon} \cancel{- \frac{\ell^2}{m r_0^3}} + \frac{3\ell^2}{m r_0^4}\epsilon \cancel{+ V'(r_0)} + V''(r_0)\epsilon = 0$$

terms cancel

$$m\ddot{\epsilon} + \left[V''(r_0) + \frac{3\ell^2}{m r_0^4}\right]\epsilon = 0$$

Condition for stability:

Frequency $V''(r_0) + \frac{3\ell^2}{m r_0^4} > 0$

4.6

$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$ for a central potential

(b)

Have $r \to r_0 + \delta r$, $\phi \to (\Omega + \delta\Omega)t$

$V_{eff}(r) = V(r) + \frac{l^2}{2mr^2}$, where $l = mr^2\dot{\phi}$ = constant for our case

So, $E = \frac{1}{2}m\delta\dot{r}^2 + V_{eff}(r_0 + \delta r)$, where $V_{eff}(r_0+\delta r) \simeq \underbrace{V_{eff}(r_0)}_{\text{can ignore constant by changing zero of potential}} + \cancel{\frac{dV_{eff}(r_0)}{dr}}^{\,0}\delta r + \frac{1}{2}\frac{d^2V_{eff}}{dr^2}\delta r^2$

$$E \approx \frac{1}{2}m\delta\dot{r}^2 + \frac{1}{2}V''_{eff}\,\delta r^2$$

$$\hookrightarrow \omega^2 = \boxed{\left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} > 0} \text{ for stability}$$

$V'_{eff} = 0$ for a circular orbit

$$V'_{eff} = V'(r) - \frac{L^2}{mr^3} = 0 \to \frac{dV}{dr} = \frac{L^2}{mr^3}$$

$$V''_{eff} = V'' + \frac{3}{r}\frac{dV}{dr} = \boxed{\left.\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right)\right|_{r_0} > 0}$$

(c) $V = -\lambda r^{-n}$

$V' = n\lambda r^{-n-1}$

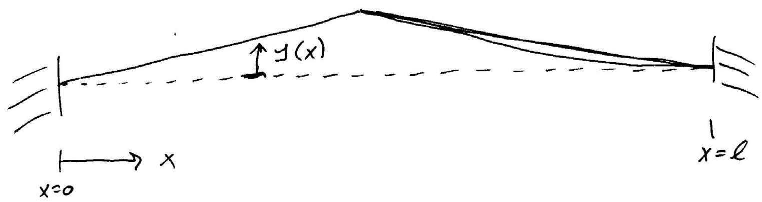
$$\frac{1}{r^3}\frac{d}{dr}\left(r^3 n\lambda r^{-n-1}\right) = \frac{1}{r^3}\frac{d}{dr}\left(n\lambda r^{-n+2}\right) = n\lambda\frac{1}{r^3}(2-n)r^{-n+1} = n\lambda(2-n)r^{-n-2} > 0$$

Given $\lambda > 0$, this is only true for $n < 2$ ✓

Tucker Elleflot

7.1

a) $y(x) = u(x,0) = \begin{cases} \dfrac{2hx}{\ell} & 0 \le x \le \frac{1}{2}\ell \\ \dfrac{2h(\ell - x)}{\ell} & \frac{1}{2}\ell \le x \le \ell \end{cases}$



y(x) can be decomposed into normal modes

$$y(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} y(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell} \int_0^{\ell/2} \frac{2hx}{\ell} \sin \frac{n\pi x}{\ell}\, dx + \frac{2}{\ell} \int_{\ell/2}^{\ell} \frac{2h(\ell - x)}{\ell} \sin \frac{n\pi x}{\ell}\, dx$$

$$\rightarrow b_n = \frac{4h}{\ell^2}\left(-\frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_0^{\ell/2} + \frac{\ell^2}{\pi^2 n^2}\sin\frac{n\pi x}{\ell}\Big|_0^{\ell/2} + \frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_{\ell/2}^{\ell} \right.$$

$$\left. - \frac{\ell^2}{n^2\pi^2}\sin\frac{n\pi x}{\ell}\Big|_{\ell/2}^{\ell} - \frac{\ell^2}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_{\ell/2}^{\ell} \right)$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \dfrac{8h}{n^2\pi^2}(-1)^{\frac{n+3}{2}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

$$du_n = \frac{1}{2}\tau\left(\frac{dy_n}{dx}\right)^2 dx = \frac{T}{2} b_n^2 \left(\frac{\pi n}{\ell}\right)^2 \cos^2\frac{n\pi x}{\ell}\, dx$$

$$U_n = \frac{T}{2} b_n^2 \left(\frac{\pi n}{\ell}\right)^2 \int_0^{\ell} \cos^2\frac{n\pi x}{\ell}\, dx$$

$$= \frac{T}{2} b_n^2 \left(\frac{\pi n}{\ell}\right)^2 \left(\frac{x}{2} + \frac{\sin\left(2\frac{n\pi x}{\ell}\right)}{4n\pi/\ell}\right)\Bigg|_0^{\ell}$$

$$= \begin{cases} \dfrac{16 T h^2}{\pi^2 \ell}\dfrac{1}{n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

b) d'Alembert's Solution:

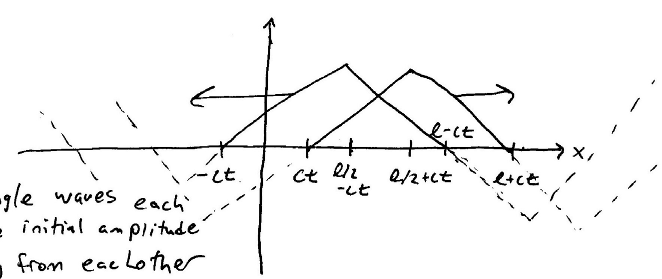
$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

Initial conditions: $u(x,0) = f(x)$

$\dot{u}(x,0) = g(x) = 0$

$$f(x-ct) = \begin{cases} \dfrac{2h\,(x-ct)}{\ell} & ct \le x \le \ell/2 + ct \\[2ex] \dfrac{2h\,(\ell - x + ct)}{\ell} & ct + \dfrac{\ell}{2} \le x \le ct + \ell \end{cases}$$

$$f(x+ct) = \begin{cases} \dfrac{2h\,(x+ct)}{\ell} & -ct \le x \le \ell/2 - ct \\[2ex] \dfrac{2h\,(\ell + ct - x)}{\ell} & \dfrac{\ell}{2} - ct \le x \le \ell - ct \end{cases}$$



two triangle waves each w/ half the initial amplitude move away from eachother

BC's $u(0,t)=0 \rightarrow f(-x) = -f(x)$

$u(\ell,t)=0 \rightarrow f[\ell + (x-\ell)] = -f[\ell-(x-\ell)]$

Physics 200A Homework 6.4

Mark Derdzinski

December 2, 2013

Problem 4 Solution

Recall the Lagrangian density for a continuous string of length L with constant density μ and tension τ clamped at both ends:

$$\mathfrak{L} = \frac{\mu}{2} y_t^2 - \frac{\tau}{2} y_x^2 \tag{1}$$

where y_t and y_x are the derivatives of the position $y(x,t)$ with respect to t or x. We want to express the Lagrangian, and eventually the Hamiltonian, in terms of fourier coefficients. We will expand $y(x,t)$ in the (complete) basis of spatial eigenfunctions

$$y(x,t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \phi_n) = \sum_{n=1}^{\infty} A_n(t) \rho_n(x) \tag{2}$$

Where the spatial eigenfunctions $\rho_n(x)$ are given by

$$\rho_n(x) = \sum_{n=1}^{\infty} \left(\frac{2}{L\mu}\right)^{\frac{1}{2}} \sin(k_n x) \tag{3}$$

Note our spatial eigenfunctions satisfy the orthonormality condition

$$\int_0^L \rho_n(x) \rho_m(x) \sigma dx = \delta_{nm} \tag{4}$$

We are now equipped to describe the system in terms of the time-dependent fourier coefficients. Suppressing x and t for clarity, the Lagrangian density expanded in fourier series becomes

$$\mathfrak{L} = \frac{\mu}{2} \left[\sum_{n=1}^{\infty} \dot{A}_n \rho_n\right] \left[\sum_{m=1}^{\infty} \dot{A}_m \rho_m\right] - \frac{\tau}{2} \left[\sum_{n=1}^{\infty} A_n \frac{d\rho_n}{dx}\right] \left[\sum_{m=1}^{\infty} A_m \frac{d\rho_m}{dx}\right] \tag{5}$$

If we integrate over x to find the full Lagrangian, we can exploit the orthonormality of the ρ_n to simplify the product of sums:

$$L(A_n, \dot{A}_n, t) = \int_0^L \mathfrak{L} dx = \frac{1}{2} \sum_{n=1}^{\infty} \left[\dot{A}_n^2 - \frac{\tau k_n^2}{\mu} A_n^2\right] = \frac{1}{2} \sum_{n=1}^{\infty} \left[\dot{A}_n^2 - \omega_n^2 A_n^2\right] \tag{6}$$

Where we have used the fact $\frac{\tau k_n^2}{\mu} = c^2 k_n^2 = w_n^2$. We are now equipped to find the Hamiltonian in terms of the fourier coefficients. The generalized momentum is

$$\pi_n = \frac{dL}{d\dot{A}_n} = \dot{A}_n \tag{7}$$

And so the Hamiltonian is given by

$$H(A_n, \dot{A}_n, t) = \sum_n^\infty \pi_n \dot{A}_n - L = \frac{1}{2}\sum_{n=1}^\infty \left[\pi_n^2 + \omega_n^2 A_n^2\right] \tag{8}$$

The Hamiltonian EOM for the fourier coefficients is now simply

$$\begin{aligned} \dot{\pi}_n &= -\frac{dH}{dA_n} \\ \implies \ddot{A}_n &= -\omega_n^2 A_n \end{aligned} \tag{9}$$

Note the Hamiltonian can be expressed in terms of

$$\begin{aligned} a_n^+ &= \pi_n + \imath\omega_n A_n \\ a_n^- &= \pi_n - \imath\omega_n A_n \\ \implies H &= \frac{1}{2}\sum_n^\infty a_n^+ a_n^- \end{aligned} \tag{10}$$

The a_n^+ and a_n^- are analogous to creation and annihilation operators for our purely classical system, where the zero-point energy is vanishing due to the classical commutator $[\pi_n, A_n] = 0$.

5) $\cancel{\mathcal{L}} \quad \frac{1}{2}\sigma \dot{f}^2 = T$ if $f(x,y)$ describes the surface

$$\mathcal{U} = T\,\delta A = T(dS - dA). \qquad dA = \hat{z}\cdot(\hat{n}\,dS)$$

normal vector of the surface element

where $\hat{n}$ is given by $\dfrac{\nabla(z - f(x,y))}{|\;''\;\;''\;|} = \dfrac{\hat{z} - \nabla f}{\sqrt{1+(\nabla f)^2}}$

$$\Rightarrow \quad dA = \frac{dS}{\sqrt{1+(\nabla f)^2}} \quad \Longrightarrow \quad \mathcal{U}_{\text{density}} = T\left(\sqrt{1+(\nabla f)^2} - 1\right)$$

$\rightarrow \quad \mathcal{L} = \frac{1}{2}\sigma \dot{f}^2 - T\left(\sqrt{1+(\nabla f)^2} - 1\right)$ and Lagrange eqns yield

$$\sigma f_{tt} = \nabla\cdot\left(\frac{T\,\nabla f}{\sqrt{1+(\nabla f)^2}}\right) \qquad \text{nonlinear wave eq.}$$

for small oscillations, this reduces to

$$\underline{\sigma f_{tt} = T\,\nabla^2 f}$$

now consider $\mathcal{H} = \pi \dot{f} - \mathcal{L}$; $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{f}}$

$$\frac{d\mathcal{H}}{dt} = \pi \ddot{f} + \dot{\pi}\dot{f} - \frac{d\mathcal{L}}{dt} = \cancel{\pi\ddot{f}} + \dot{\pi}\dot{f} - \left(\frac{\partial \mathcal{L}}{\partial f}\dot{f} + \cancel{\frac{\partial \mathcal{L}}{\partial \dot{f}}\ddot{f}} + \frac{\partial \mathcal{L}}{\partial \nabla f}(\dot{\nabla f})\right)$$

$$\rightarrow \quad \dot{\pi}\dot{f} - \underbrace{\frac{\partial \mathcal{L}}{\partial f}}_{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{f}} + \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla f}}\dot{f} - \frac{\partial \mathcal{L}}{\partial \nabla f}(\dot{\nabla f}) = \cancel{\dot{\pi}\dot{f} - \dot{\pi}\dot{f}} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla f}\dot{f} - \frac{\partial \mathcal{L}}{\partial \nabla f}(\nabla\dot{f})$$

$$\Rightarrow \quad \frac{d\mathcal{H}}{dt} = -\nabla\cdot\frac{\partial\mathcal{L}}{\partial f}\dot{f} - \frac{\partial\mathcal{L}}{\partial f}\nabla(\dot{f}) = \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla f}\dot{f}\right)$$

$$\Rightarrow \quad \frac{d\mathcal{H}}{dt} + \nabla\cdot\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\nabla f}\dot{f}\right)}_{\equiv S} = 0$$

Energy conservation

$$\frac{dH}{dt} + \nabla\cdot S = 0$$

for momentum conservation, consider $\frac{dS}{dt}$

let $P_{wave} = \frac{S}{v_{ph}^2}$, $v_{ph} \equiv \frac{T}{\sigma}$. $\frac{dP}{dt} = \frac{1}{v_{ph}^2}\frac{dS}{dt}$

$$= \frac{\sigma}{T}\left[\partial_t\left(\frac{\partial\mathcal{L}}{\partial\nabla f}\right)\dot{f} + \ddot{f}\frac{\partial\mathcal{L}}{\partial\nabla f}\right] \; ; \text{ use } \ddot{f} = \frac{T}{\sigma}\nabla^2 f \;\&\; \frac{\partial\mathcal{L}}{\partial\nabla f} = -T\nabla f$$

$$\Rightarrow \quad \frac{\sigma}{T}\left[-T\,\partial_t(\nabla f)\dot{f} - T\nabla f\,\frac{T}{\sigma}\nabla^2 f\right]$$

$$= -\left[\nabla f\,\nabla^2 f + \sigma\dot{f}\,\nabla\dot{f}\right] = -\nabla\cdot\left[\frac{T(\nabla f)^2}{2} + \frac{\sigma(\dot{f})^2}{2}\right]$$

so we have $\underline{\varepsilon} \equiv \cancel{\frac{T(\nabla f)^2}{2} + \frac{\sigma\dot{f}^2}{2}}$

$$\Rightarrow \quad \underline{\partial_t P_{wave} + \nabla\cdot\underline{\varepsilon} = 0}$$

momentum conservation

$$\cancel{\underline{\varepsilon} = \frac{\sigma\dot{f}^2}{2} + \frac{1}{2}\begin{pmatrix}\partial^2 f/\partial x^2 \\ \partial^2 f/\partial y^2\end{pmatrix}} \quad \cancel{P_{wave} = \begin{pmatrix}P_x \\ P_y\end{pmatrix}}$$

~~act~~ actually, have to be a little more careful defining ε

$$S_x = f_t \frac{\partial \mathcal{L}}{\partial f_x} \quad \text{so} \quad \frac{d}{dt} S_x = f_{tt} \frac{\partial \mathcal{L}}{\partial f_x} + f_t \partial_t \frac{\partial \mathcal{L}}{\partial f_x}$$

$$\frac{d}{dt} S_x = \frac{T}{\sigma}\left(f_{xx} + f_{yy}\right) - T f_x + f_t\left(-T f_{xt}\right)$$

$$= -\frac{T}{\sigma}\left(T\left(f_{xx} f_x + f_{yy} f_x\right) + \sigma f_t f_{xt}\right)$$

$$= -\frac{T}{\sigma}\left(\partial_x\left(\frac{f_x^2}{2} T + \frac{\sigma f_t^2}{2}\right) + T f_{yy} f_x\right)$$

$$= \underline{-\frac{T}{\sigma}\left[\frac{\partial}{\partial x}\left(\frac{T f_x^2}{2} + \frac{\sigma f_t^2}{2} - \frac{T f_y^2}{2}\right) + \frac{\partial}{\partial y}\left(T f_y f_x\right)\right]}$$

similarly, $\frac{dS_y}{dt} = -\frac{T}{\sigma}\left[\frac{\partial}{\partial y}\left(\frac{T f_y^2}{2} + \frac{\sigma f_t^2}{2} - \frac{T f_x^2}{2}\right) + \frac{\partial}{\partial x}\left(T f_x f_y\right)\right]$

$$\rightarrow \quad \underline{\varepsilon} = \begin{pmatrix} T\left[\frac{f_x^2}{2} - \frac{f_y^2}{2}\right] + \frac{\sigma f_t^2}{2} & T f_x f_y \\ T f_x f_y & \frac{T}{2}\left[f_y^2 - f_x^2\right] + \frac{\sigma f_t^2}{2} \end{pmatrix}$$

& have ~~st~~ $\quad \partial_t P_{wave} + \underbrace{\nabla}_{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)} \cdot \varepsilon = 0$

And if we want we can write

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{where} \quad \partial_\mu = \left(\frac{1}{v_{ph}} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{H} & S_x/v_{ph} & S_y/v_{ph} \\ S_x/v_{ph} & \varepsilon_{xx} & \varepsilon_{xy} \\ S_y/v_{ph} & \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix}$$