Aleksandra Shirman

Homework 2

1. a) $v(x,y) = \dfrac{c}{n(x,y)}$

$$T = \int_1^2 dt = \frac{1}{c}\int \frac{c}{v}\frac{ds}{dt}\,dt = \frac{1}{c}\int n(x,y)\,ds$$

$$= \frac{1}{c}\int n(y(x))\,dx\sqrt{1+y'^2} \qquad y' = \frac{dy}{dx}$$

$$= \frac{1}{c}\int dx\,\underbrace{n(y)\sqrt{1+y'^2}}_{\equiv \mathcal{L}}$$

$$\delta T = 0 = \delta\frac{1}{c}\int dx\,\mathcal{L} = \frac{1}{c}\int dx\,\delta\mathcal{L}$$

$$= \frac{1}{c}\int\left[\frac{\partial\mathcal{L}}{\partial y}\delta y + \frac{\partial\mathcal{L}}{\partial y'}\delta y'\right]dx$$

$$= \frac{1}{c}\int\left[\frac{\partial n}{\partial y}(1+y'^2)^{1/2}\delta y + n(1+y'^2)^{-1/2}y'\delta y'\right]dx$$

$$= \frac{1}{c}\int\left[\frac{\partial n}{\partial y}(1+y'^2)^{1/2}\delta y - \frac{d}{dx}\left[n(1+y'^2)^{-1/2}y'\right]\delta y\right]dx + \underbrace{n(1+y'^2)^{-1/2}y'\delta y}_{\text{at endpoints}\ \to\ 0}$$

$$= \frac{1}{c}\int dx\,\delta y\left[\frac{\partial}{\partial y}\left(n(1+y'^2)^{1/2}\right) - \frac{d}{dx}\frac{\partial}{\partial y'}\left(n(1+y'^2)^{1/2}\right)\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial y}\left[n(1+y'^2)^{1/2}\right] - \frac{d}{dx}\frac{\partial}{\partial y'}\left[n(1+y'^2)^{1/2}\right] = 0$$

gives the path $y(x)$ for shortest time

b) $\frac{\partial}{\partial x}\left[n(1+x'^2)^{1/2}\right] - \frac{d}{dy}\frac{\partial}{\partial x'}\left[n(1+x'^2)^{1/2}\right] = 0$ for $x' = \frac{dx}{dy}$

is the same thing as above

for constant n:

$$0 - \frac{d}{dy}\left[\underbrace{n(1+x'^2)^{-1/2}x'}_{\text{constant value}}\right] = 0$$

constant value → constant slope
⇒ straight line

now using

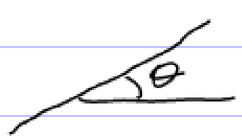
$$\frac{\partial}{\partial y}\left[n(1+y'^2)^{1/2}\right] - \frac{d}{dx}\frac{\partial}{\partial y'}\left[n(1+y'^2)^{1/2}\right] = 0$$

$n = n(x)$ does not depend on y so first term is 0

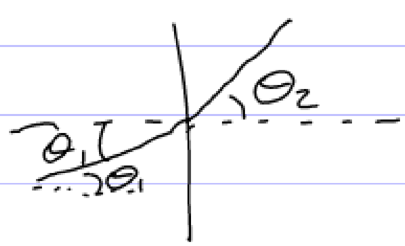
$$0 - \frac{d}{dx}\underbrace{\frac{\partial}{\partial y'}\left[n(1+y'^2)^{1/2}\right]}_{\text{constant}} = 0$$

$$\frac{\partial}{\partial y'}\left[n(1+y'^2)^{1/2}\right] = \frac{ny'}{\sqrt{1+y'^2}} = \frac{n\,dy}{\sqrt{dx^2+dy^2}} = n\sin\theta$$

where θ is defined as the angle above the horizontal



⇒ since true for all x,
⇒ $n_1\sin\theta_1 = n_2\sin\theta_2$
which is snell's law
QED



Physics 200A Homework 2.2

Mark Derdzinski

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Problem 2 Solution

Part (a)

We begin with the Lagrangian density:

$$\mathfrak{L} = -\frac{\hbar^2}{2m}\left(\frac{d\Psi^*}{dx}\right)\left(\frac{d\Psi}{dx}\right) - \Psi^*(U-E)\Psi \ . \tag{1}$$

The Euler-Lagrange Equation corresponding to Ψ^* is given by

$$\begin{aligned}
&\frac{d}{dx}\frac{\partial \mathfrak{L}}{\partial \frac{d\Psi^*}{dx}} - \frac{\partial \mathfrak{L}}{\partial \Psi^*} = 0 \\
\implies &\frac{d}{dx}\left(-\frac{\hbar^2}{2m}\frac{d\Psi}{dx}\right) - (-\Psi(U-E)) = 0 \\
\implies &-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + U\Psi = E\Psi
\end{aligned} \tag{2}$$

Thus we recover the SE for Ψ. $\square$

Part (b)

Consider the probability $P = \Psi^*\Psi$. It is invariant under the following symmetry:

$$\begin{aligned}
\Psi &\to e^{\frac{\imath\theta}{\hbar}}\Psi \\
\Psi^* &\to e^{\frac{-\imath\theta}{\hbar}}\Psi^*
\end{aligned}$$

Such a complex phase rotation preserves the Lagrangian, so we can apply Noether's Theorem. In the infitesimal limit where $\theta \to \delta\theta$,

$$\begin{aligned}
\Psi &\to \Psi + \frac{\imath}{\hbar}\Psi(\delta\theta) = \Psi + (\delta\theta)Q_\Psi \\
\Psi^* &\to \Psi^* - \frac{\imath}{\hbar}\Psi^*(\delta\theta) = \Psi^* + (\delta\theta)Q_{\Psi^*}
\end{aligned}$$

Where we identify the generators of our coordinates,

$$
\begin{aligned}
Q_\Psi &= \frac{\imath}{\hbar}\Psi \\
Q_{\Psi^*} &= -\frac{\imath}{\hbar}\Psi^*
\end{aligned}
$$

By Noether's Theorem, the quantity

$$
\begin{aligned}
\jmath &= -\left(\frac{\partial \mathfrak{L}}{\partial \frac{d\Psi}{dx}}\right) Q_\Psi - \left(\frac{\partial \mathfrak{L}}{\partial \frac{d\Psi^*}{dx}}\right) Q_{\Psi^*} \\
&= \frac{\imath\hbar}{2m}\frac{d\Psi^*}{dx}\Psi - \frac{\imath\hbar}{2m}\frac{d\Psi}{dx}\Psi^* \\
&= \frac{\hbar}{2m\imath}\left[\Psi^*\frac{d\Psi}{dx} + \frac{d\Psi^*}{dx}\Psi\right] \qquad (3)
\end{aligned}
$$

which we identify as the *probability current*, is conserved (i.e. $\frac{\partial \jmath}{\partial x} = 0$).
We now return to the time derivative of probability:

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{\partial(\Psi^*\Psi)}{\partial t} \\
&= \frac{\partial \Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial \Psi}{\partial t} \qquad (4)
\end{aligned}
$$

Using the explicit expression for $E = \imath\hbar\frac{\partial}{\partial t}$ in the SE, we can rearrange Eqn. 2 to solve for $\frac{\partial \Psi}{\partial t}$:

$$
\frac{d\Psi}{dt} = \frac{1}{\imath\hbar}\left[-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + U\Psi\right] \qquad (5)
$$

If $U \in \Re$, plugging this and the conjugate equation into Eqn. 4 yields the continuity equation:

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{1}{\imath\hbar}\left[\frac{\hbar^2}{2m}\frac{d^2\Psi^*}{dx^2}\Psi - U\Psi^*\Psi - \frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}\Psi^* + U\Psi^*\Psi\right] \\
&= \frac{\hbar}{2m\imath}\left[\frac{d^2\Psi^*}{dx^2}\Psi - \Psi^*\frac{d^2\Psi}{dx^2}\right] \\
&= \frac{\hbar}{2m\imath}\frac{d}{dx}\left[\frac{d\Psi^*}{dx}\Psi - \Psi^*\frac{d\Psi}{dx}\right] = -\frac{d\jmath}{dx} \qquad (6)
\end{aligned}
$$

Thus the time derivative of probability is equal to the divergence of the probability current, which, by Noether's theorem, is zero. □

(3) $\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)\, dt = 0$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} \right) dt = 0$$

Integrate by parts: (recall $\int_a^b f g' = -\int_a^b f' g + fg \Big|_a^b$)

$$0 = \underbrace{\left[\frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \right]_{t_1}^{t_2}}_{0 \text{ if } \delta \dot{q}(t_{1,2}) = 0} + \underbrace{\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2}}_{0\text{, as usual, since } \delta q(t_1) = \delta q(t_2) = 0} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta \dot{q} \right] dt$$

So, integrate by parts again:

$$0 = \underbrace{\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) \right] \delta q\, dt}_{\text{Must } = 0 \text{ for all } \delta q(t)} + \underbrace{\left[\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \delta q \right]_{t_1}^{t_2}}_{0}$$

Therefore, $\boxed{\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) + \dfrac{d^2}{dt^2}\left(\dfrac{\partial L}{\partial \ddot{q}_i}\right) = 0}$

(b) $$\frac{dL}{dt} = \sum \frac{\partial L}{\partial q_i}\dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i}\ddot{q} + \sum \frac{\partial L}{\partial \ddot{q}}\dddot{q} + \overbrace{\left(\frac{\partial L}{\partial t}\right)}^{0}$$

In normal case, use Lagrange eqn. to eliminate $\frac{\partial L}{\partial q_i}$:

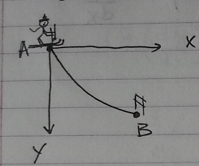
$$\frac{dL}{dt} = \sum \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i}\right)\dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \sum \frac{\partial L}{\partial \ddot{q}_i}\dddot{q}_i$$

$$= \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \sum \frac{d}{dt}\left(\ddot{q}_i \frac{\partial L}{\partial \ddot{q}_i} - \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i}\right)\Big]$$

$$\Rightarrow \frac{d}{dt}\Big[\underbrace{\sum \left\{\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \ddot{q}_i \frac{\partial L}{\partial \ddot{q}_i} - \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i}\right\} - L}_{\text{"Energy", which is conserved.}}\Big] = 0$$

Phys 200, HW 2 Petia Yanchulova

#4



The time to travel from A to B is

$$t_{AB} = \int_A^B \frac{ds}{v}$$

From ENERGY conservation

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = (1 + y'(x)^2)^{1/2}\,dx$$

Then, $$t_{AB} = \int_A^B \frac{(1+y'^2)^{1/2}}{\sqrt{2gy}}\,dx$$

What is the function, or path, which makes the $\int$ above a minimum?

The functional (fn to be varied, a fn of the path) is:

$$F = \left[\frac{1+y'^2}{2gy}\right]^{1/2}$$

If t is to be a minimum, the fn y(x) must satisfy the eqn:

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

This follows from the E-L eqn for the variational problem (E-L differential eqn) on pp 62-64 F&W.

F has no explicit dependence on x, and if this is the case, then the E-L DE can be expressed as:

2.4.2

mult. by y': $\left[\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right] y' = 0$ $y' = \frac{dy}{dx}$

$$\frac{dy}{dx}\cdot\frac{d}{dx}\frac{\partial F}{\partial y'} - \cancel{\frac{dy}{dx}}\frac{\partial F}{\cancel{\partial y}} = 0$$

$$y'\frac{d}{dx}\frac{\partial F}{\partial y'} + y''\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial x} = 0$$

$$\frac{d}{dx}\left[y'\frac{\partial F}{\partial y'} - F\right] = 0 \quad \text{or} \quad y'\frac{\partial F}{\partial y'} - F = \text{constant}$$

So $y'\frac{\partial F}{\partial y'} - F = C$ $F = \left[\frac{1+y'^2}{2gy}\right]^{1/2}$

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\left[\frac{1+y'^2}{2gy}\right]^{1/2} = \frac{1}{(2gy)^{1/2}}\cdot\frac{1}{2}\left[1+y'^2\right]^{-1/2}\cdot 2y'$$

$$= \frac{y'}{(2gy)^{1/2}(1+y'^2)^{1/2}}$$

$$\frac{y'^2}{(2gy)^{1/2}(1+y'^2)^{1/2}} - \frac{(1+y'^2)^{1/2}}{(2gy)^{1/2}} = C$$

square: both sides

$$\left[\frac{\cancel{y'^2} - 1 - \cancel{y'^2}}{(2gy)^{1/2}(1+y'^2)^{1/2}}\right]^2 = \left[C\right]^2$$

$$\frac{1}{2gy(1+y'^2)} = C^2$$

or $\underline{y(1+y'^2) = a}$ where $a = \frac{1}{C^2 2g}$

2.4.3

ut $y(1+y'^2) = a$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a}{y}$$

$$\frac{dy}{dx} = \sqrt{\frac{a}{y} - 1}$$

$$dx = \frac{dy}{\sqrt{\frac{a}{y} - 1}} \Rightarrow x = \int \left(\frac{a}{y} - 1\right)^{-1/2} dy$$

Let $y = a \sin^2 \frac{\theta}{2}$ $\left(= \frac{a}{2}(1 - \cos\theta)\right)$

$$dy = 2a \sin\frac{\theta}{2} \frac{1}{2} \cos\frac{\theta}{2} \, d\theta = a \sin\frac{\theta}{2} \cos\frac{\theta}{2} \, d\theta$$

$$x = \int \frac{\left(a \sin^2 \frac{\theta}{2}\right)^{1/2} . \, a \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{\underbrace{\left(a - a\sin^2\frac{\theta}{2}\right)^{1/2}}_{a(1 - \sin^2\frac{\theta}{2}) \, = \, a(\cos^2\frac{\theta}{2})}} d\theta$$

$$= \int \frac{a^{1/2} \sin\theta/2 . \, a \sin\theta/2 \, \cancel{\cos\theta/2}}{a^{1/2} \cancel{(\cos^2\theta/2)^{1/2}}} d\theta = a \int \sin^2 \frac{\theta}{2} \, d\theta$$

$$= \frac{a}{2} \int (1 - \cos\theta) \, d\theta$$

$x = \frac{a}{2}(\theta - \sin\theta) + C$. Setting the origin at $x=0$, $y=0$, $\theta = 0$ gives:

$$\boxed{\begin{aligned} x &= \frac{a}{2}(\theta - \sin\theta) \\ y &= \frac{a}{2}(1 - \cos\theta) \end{aligned}}$$

5

Consider a system with Lagrangian in Cartesian coordinates which is given by:

$$L = \frac{1}{2}\sum_a m_a\left(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2\right) - U$$

Now, if this is more conveniently expressed in terms of generalized coordinates $(q_1, q_2, \ldots, q_s)$, use the transformations:

$$\mathbf{x} = \mathbf{f}(q_1, q_2, \ldots, q_s)$$
$$\dot{\mathbf{x}} = \sum_c \frac{\partial \mathbf{f}}{\partial q_c}\dot{q}_c$$

(a) What is the most general form of L in terms of the q's?
T is given in general by

$$T = \frac{1}{2}\sum_a m_a\left(\sum_b \frac{\partial \mathbf{f}_a}{\partial q_b}\dot{b}_c\right)^2, \tag{1}$$

where the sums a and b are over the number of particles and generalized coordinates respectively. $U(\mathbf{x})$ simply becomes $U(\mathbf{f})$.

(b) Specify the form of the kinetic energy as fully as possible.
If $\mathbf{f}$ does not depend on time explicitly, the expression for T in generalized coordinates will have the form

$$T = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k. \tag{2}$$

A comparison between (1) and (2) shows that

$$M_{jk} = \sum_a m_a \frac{\partial \mathbf{f}_a}{\partial q_j}\cdot\frac{\partial \mathbf{f}_a}{\partial q_k}. \tag{3}$$

Note also that M_{jk} is symmetric in its indicies.

(c) Derive the Lagrangian EOMs

$$\frac{\partial L}{\partial q_m} = -\sum_a \frac{\partial \mathbf{f}_a}{\partial q_m}\cdot\frac{\partial U}{\partial \mathbf{f}_a} \equiv Q_m$$
$$\frac{\partial L}{\partial \dot{q}_m} = M_{jm}\dot{q}_j$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} = M_{jm}\ddot{q}_j + \frac{dM_{jm}}{dt}\dot{q}_j$$

where dM_{jm}/dt is can be obtained using the chain rule. The EOM is then

$$M_{jm}\ddot{q}_j + \frac{dM_{jm}}{dt}\dot{q}_j = Q_m \tag{4}$$