

Formulas:

Time dilation; Length contraction: $\Delta t = \gamma \Delta t' \equiv \gamma \Delta t_p$; $L = L_p / \gamma$; $c = 3 \times 10^8 \text{ m/s}$

Lorentz transformation: $x' = \gamma(x - vt)$; $y' = y$; $z' = z$; $t' = \gamma(t - vx/c^2)$; inverse: $v \rightarrow -v$

Velocity transformation: $u_x' = \frac{u_x - v}{1 - u_x v/c^2}$; $u_y' = \frac{u_y}{\gamma(1 - u_x v/c^2)}$; inverse: $v \rightarrow -v$

Spacetime interval: $(\Delta s)^2 = (c\Delta t)^2 - [\Delta x^2 + \Delta y^2 + \Delta z^2]$

Relativistic Doppler shift: $f_{obs} = f_{source} \sqrt{1 + v/c} / \sqrt{1 - v/c}$

Momentum: $\vec{p} = \gamma m \vec{u}$; Energy: $E = \gamma mc^2$; Kinetic energy: $K = (\gamma - 1)mc^2$

Rest energy: $E_0 = mc^2$; $E = \sqrt{p^2 c^2 + m^2 c^4}$

Electron: $m_e = 0.511 \text{ MeV}/c^2$; Proton: $m_p = 938.26 \text{ MeV}/c^2$; Neutron: $m_n = 939.55 \text{ MeV}/c^2$

Atomic mass unit: $1 u = 931.5 \text{ MeV}/c^2$; electron volt: $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$

Stefan's law: $e_{tot} = \sigma T^4$, e_{tot} = power/unit area ; $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \text{K}^4$

$e_{tot} = cU/4$, U = energy density = $\int_0^\infty u(\lambda, T) d\lambda$; Wien's law: $\lambda_m T = \frac{hc}{4.96 k_B}$

Boltzmann distribution: $P(E) = C e^{-E/(k_B T)}$

Planck's law: $u_\lambda(\lambda, T) = N_\lambda(\lambda) \times \bar{E}(\lambda, T) = \frac{8\pi}{\lambda^4} \times \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1}$; $N(f) = \frac{8\pi f^2}{c^3}$

Photons: $E = hf = pc$; $f = c/\lambda$; $hc = 12,400 \text{ eV \AA}$; $k_B = (1/11,600) \text{ eV/K}$

Photoelectric effect: $eV_s = K_{max} = hf - \phi$, ϕ = work function; Bragg equation: $n\lambda = 2d \sin \theta$

Compton scattering: $\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$; $\frac{h}{m_e c} = 0.0243 \text{ \AA}$

Coulomb force: $F = \frac{kq_1 q_2}{r^2}$; Coulomb energy: $U = \frac{kq_1 q_2}{r}$; Coulomb potential: $V = \frac{kq}{r}$

Force in electric and magnetic fields (Lorentz force): $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

Rutherford scattering: $\Delta n = C \frac{Z^2}{K_\alpha^2} \frac{1}{\sin^4(\phi/2)}$; $ke^2 = 14.4 \text{ eV \AA}$; $\hbar c = 1973 \text{ eV \AA}$

Hydrogen spectrum: $\frac{1}{\lambda_{mn}} = R \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$; $R = 1.097 \times 10^7 \text{ m}^{-1} = \frac{1}{911.3 \text{ \AA}}$

Bohr atom: $E_n = -\frac{ke^2 Z}{2r_n} = -E_0 \frac{Z^2}{n^2}$; $E_0 = \frac{ke^2}{2a_0} = \frac{m_e (ke^2)}{2\hbar^2} = 13.6 \text{ eV}$; $K = \frac{m_e v^2}{2}$; $U = -\frac{ke^2 Z}{r}$

$hf = E_i - E_f$; $r_n = r_0 n^2$; $r_0 = \frac{a_0}{Z}$; $a_0 = \frac{\hbar^2}{m_e ke^2} = 0.529 \text{ \AA}$; $L = m_e v r = n\hbar$ angular momentum

de Broglie: $\lambda = \frac{h}{p}$; $f = \frac{E}{h}$; $\omega = 2\pi f$; $k = \frac{2\pi}{\lambda}$; $E = \hbar \omega$; $p = \hbar k$; $E = \frac{p^2}{2m}$

Wave packets: $y(x, t) = \sum_j a_j \cos(k_j x - \omega_j t)$, or $y(x, t) = \int dk a(k) e^{i(kx - \omega(k)t)}$; $\Delta k \Delta x \sim 1$; $\Delta \omega \Delta t \sim 1$

group and phase velocity : $v_g = \frac{d\omega}{dk}$; $v_p = \frac{\omega}{k}$; Heisenberg: $\Delta x \Delta p \sim \hbar$; $\Delta t \Delta E \sim \hbar$

Schrodinger equation: $-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi(x, t) = i\hbar \frac{\partial \Psi}{\partial t}$; $\Psi(x, t) = \psi(x) e^{-i\frac{E}{\hbar} t}$

Time-independent Schrodinger equation: $-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U(x)\psi(x) = E\psi(x)$; $\int_{-\infty}^{\infty} dx \psi^* \psi = 1$

∞ square well: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$; $E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}$; $\frac{\hbar^2}{2m_e} = 3.81 \text{ eV}\text{\AA}^2$ (electron)

Harmonic oscillator: $\Psi_n(x) = H_n(x) e^{-\frac{m\omega}{2\hbar} x^2}$; $E_n = (n + \frac{1}{2})\hbar\omega$; $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$; $\Delta n = \pm 1$

Expectation value of $[Q]$: $\langle Q \rangle = \int \psi^*(x)[Q]\psi(x) dx$; Momentum operator: $p = \frac{\hbar}{i} \frac{\partial}{\partial x}$

Eigenvalues and eigenfunctions: $[Q]\Psi = q\Psi$ (q is a constant); uncertainty: $\Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$

Step potential: reflection coef: $R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$, $T = 1 - R$; $k = \sqrt{\frac{2m}{\hbar^2}(E - U)}$

Tunneling: $\psi(x) \sim e^{-\alpha x}$; $T = e^{-2\alpha \Delta x}$; $T = e^{-2 \int_{x_1}^{x_2} \alpha(x) dx}$; $\alpha(x) = \sqrt{\frac{2m[U(x) - E]}{\hbar^2}}$

Schrodinger equation in 3D: $-\frac{\hbar^2}{2m} \nabla^2 \Psi + U(\vec{r})\Psi(\vec{r}, t) = i\hbar \frac{\partial \Psi}{\partial t}$; $\Psi(\vec{r}, t) = \psi(\vec{r}) e^{-i\frac{E}{\hbar} t}$

3D square well: $\Psi(x, y, z) = \Psi_1(x)\Psi_2(y)\Psi_3(z)$; $E = \frac{\pi^2 \hbar^2}{2m} \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right)$

Spherically symmetric potential: $\Psi_{n, \ell, m_\ell}(r, \theta, \phi) = R_{n\ell}(r) Y_\ell^{m_\ell}(\theta, \phi)$; $Y_\ell^{m_\ell}(\theta, \phi) = P_\ell^{m_\ell}(\theta) e^{im_\ell \phi}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$; $[L_z] = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$; $[L^2] Y_\ell^{m_\ell} = \ell(\ell + 1)\hbar^2 Y_\ell^{m_\ell}$; $[L_z] Y_\ell^{m_\ell} = m_\ell \hbar Y_\ell^{m_\ell}$

Radial probability density: $P(r) = r^2 |R_{n\ell}(r)|^2$; Energy: $E_n = -\frac{ke^2}{2a_0} \frac{Z^2}{n^2}$

Ground state of hydrogen and hydrogen-like ions: $\Psi_{1,0,0} = \frac{1}{\pi^{1/2}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$

Justify all your answers to all problems. Write clearly.

Problem 1 (10 points)

The ground state wavefunction for a particle in a harmonic oscillator potential is

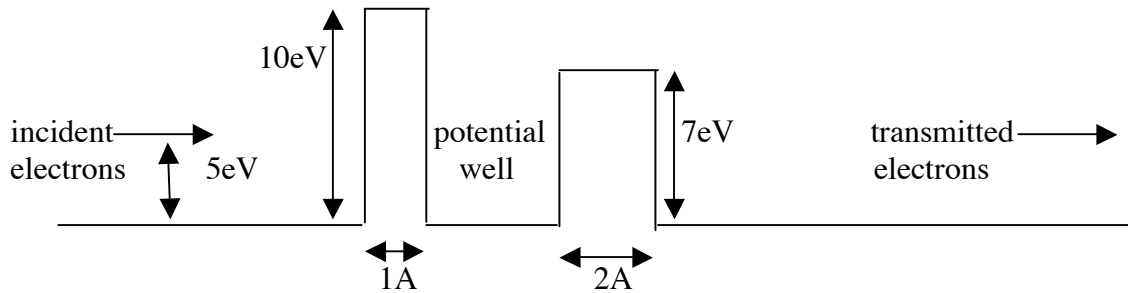
$\Psi(x) = D e^{-\beta x^2}$ with $\beta = 4\text{\AA}^{-2}$ and D a constant.

- Find an expression for D in terms of β .
- Find the uncertainty in the position of this particle, Δx , in Angstrom.
- Find an expression for the uncertainty in the momentum of this particle, Δp , expressed in terms of β and \hbar .
- If this particle has mass $1973 \text{ eV}/c^2$, find its average kinetic energy in eV.

Use $\int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$ and $\int_{-\infty}^{\infty} dx x^2 e^{-\lambda x^2} = \frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}}$. Do not use any of the formulas for the

harmonic oscillator potential, do not use the symbol ω in any of your expressions, and show all steps in your calculations.

Problem 2 (10 points)



Electrons with energy 5eV are incident from the left on a barrier of width 1A and height 10eV. If an electron can go through the barrier it finds a potential well where it stays for a long time bouncing back and forth (assume its energy doesn't change). It can then escape from the well through the barrier it came in, or through the barrier on the right side of height 7eV and width 2A.

- What is approximately the probability that an electron incident from the left will be transmitted through the first barrier and enter the potential well?
- For an electron that made it into the potential well, is it more likely to come out through the right barrier or through the left barrier? Estimate the ratio of the probabilities (right/left).
- Using the results of (a) and (b), estimate how many transmitted electrons will result for 3000 incident electrons, where "transmitted electrons" are those that went through both barriers and continue to the right (see figure). From this, deduce the overall transmission (= # transmitted / # incident) and reflection (# reflected / # incident) coefficients for this composite barrier. (Note that the sum of both coefficients should be 1).

Use $\frac{\hbar^2}{2m_e} = 3.81eVA^2$.

Problem 3 (10 points)

The wavefunction for an electron in a hydrogen-like ion with $Z=2$ is

$$\psi(r, \theta, \phi) = C r e^{-r/a_0} \sin \theta e^{-i\phi}$$

- Give the values of the quantum numbers n , ℓ , m_ℓ . Justify your answers.
- Give the value of the z-component of the angular momentum.
- Give the value of the most probable distance of this electron to the nucleus, expressed in terms of a_0 . Show your work. Compare your answer with the radius of the orbit in the Bohr model with the same n and Z .

Justify all your answers to all problems. Write clearly.