

5-26 The full width at half-maximum (FWHM) is 110 MeV. So $\Delta E = 55 \text{ MeV}$ and using

$$\Delta E_{\min} \Delta t_{\min} = \frac{\hbar}{2},$$

$$\Delta t_{\min} = \frac{\hbar}{2\Delta E} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{2(55 \times 10^6 \text{ eV})} \cong 6.0 \times 10^{-24} \text{ s}$$

$$\tau = \text{lifetime} \sim 2\Delta t_{\min} = 1.2 \times 10^{-23} \text{ s}$$

5-27 For a single slit with width a , minima are given by $\sin \theta = \frac{n\lambda}{a}$ where $n = 1, 2, 3, \dots$ and

$$\sin \theta \approx \tan \theta = \frac{x}{L}, \quad \frac{x_1}{L} = \frac{\lambda}{a} \quad \text{and} \quad \frac{x_2}{L} = \frac{2\lambda}{a} \Rightarrow \frac{x_2 - x_1}{L} = \frac{\lambda}{a} \quad \text{or}$$

$$\lambda = \frac{a\Delta x}{L} = \frac{5 \text{ \AA} \times 2.1 \text{ cm}}{20 \text{ cm}} = 0.525 \text{ \AA}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2(5.11 \times 10^5 \text{ eV})(0.525 \text{ \AA})^2} = 546 \text{ eV}$$

5-32

(a)

$$f = \frac{E}{h} = \frac{(1.8)(1.6 \times 10^{-19} \text{ J})}{6.63 \times 10^{-34} \text{ J} \cdot \text{s}} = 4.34 \times 10^{14} \text{ Hz}$$

$$(b) \quad \lambda = \frac{c}{f} = 691 \text{ nm}$$

$$(c) \quad \Delta E \geq \frac{\hbar}{\Delta t} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{2\pi(2 \times 10^{-6} \text{ s})}$$

$$\Delta E \geq 5.276 \times 10^{-29} \text{ J} = 3.30 \times 10^{-10} \text{ eV}$$

6-2 (a) Normalization requires

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = A^2 \int_{-L/4}^{L/4} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{A^2}{2}\right) \int_{-L/4}^{L/4} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

$$\text{so } A = \frac{2}{\sqrt{L}}.$$

$$(b) \quad P = \int_0^{L/8} |\psi|^2 dx = A^2 \int_0^{L/8} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{4}{L}\right) \left(\frac{1}{2}\right) \int_0^{L/8} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

$$= \left(\frac{2}{L}\right) \left(\frac{L}{8}\right) + \left(\frac{2}{L}\right) \left(\frac{L}{4\pi}\right) \sin\left(\frac{4\pi x}{L}\right) \Bigg|_0^{L/8} = \frac{1}{4} + \frac{1}{2\pi} = 0.409$$

6-3 (a) $A \sin\left(\frac{2\pi x}{\lambda}\right) = A \sin(5 \times 10^{10} x)$ so $\left(\frac{2\pi}{\lambda}\right) = 5 \times 10^{10} \text{ m}^{-1}$,

$$\lambda = \frac{2\pi}{5 \times 10^{10}} = 1.26 \times 10^{-10} \text{ m}.$$

$$(b) \quad p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ Js}}{1.26 \times 10^{-10} \text{ m}} = 5.26 \times 10^{-24} \text{ kg m/s}$$

$$(c) \quad K = \frac{p^2}{2m} \quad m = 9.11 \times 10^{-31} \text{ kg}$$

$$K = \frac{(5.26 \times 10^{-24} \text{ kg m/s})^2}{(2 \times 9.11 \times 10^{-31} \text{ kg})} = 1.52 \times 10^{-17} \text{ J}$$

$$K = \frac{1.52 \times 10^{-17} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 95 \text{ eV}$$

6-6

$$\psi(x) = A \cos kx + B \sin kx$$

$$\frac{\partial \psi}{\partial x} = -kA \sin kx + kB \cos kx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 A \cos kx - k^2 B \sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E - U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A \cos kx + B \sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E - U)\psi$ or

$$-k^2(A \cos kx + B \sin kx) = \left(\frac{-2mE}{\hbar^2}\right)(A \cos kx + B \sin kx).$$

Therefore $E = \frac{\hbar^2 k^2}{2m}$.

6-9 $E_n = \frac{n^2 \hbar^2}{8mL^2}$, so $\Delta E = E_2 - E_1 = \frac{3\hbar^2}{8mL^2}$

$$\Delta E = (3) \frac{(1240 \text{ eV nm}/c)^2}{8(938.28 \times 10^6 \text{ eV}/c^2)(10^{-5} \text{ nm})^2} = 6.14 \text{ MeV}$$

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV nm}}{6.14 \times 10^6 \text{ eV}} = 2.02 \times 10^{-4} \text{ nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-10 $E_n = \frac{n^2 \hbar^2}{8mL^2}$

$$\frac{\hbar^2}{8mL^2} = \frac{(6.63 \times 10^{-34} \text{ Js})^2}{8(9.11 \times 10^{-31} \text{ kg})(10^{-10} \text{ m})^2} = 6.03 \times 10^{-18} \text{ J} = 37.7 \text{ eV}$$

(a) $E_1 = 37.7 \text{ eV}$
 $E_2 = 37.7 \times 2^2 = 151 \text{ eV}$
 $E_3 = 37.7 \times 3^2 = 339 \text{ eV}$
 $E_4 = 37.7 \times 4^2 = 603 \text{ eV}$

(b) $hf = \frac{hc}{\lambda} = E_{n_i} - E_{n_f}$
 $\lambda = \frac{hc}{E_{n_i} - E_{n_f}} = \frac{1240 \text{ eV} \cdot \text{nm}}{E_{n_i} - E_{n_f}}$

For $n_i = 4$, $n_f = 1$, $E_{n_i} - E_{n_f} = 603 \text{ eV} - 37.7 \text{ eV} = 565 \text{ eV}$, $\lambda = 2.19 \text{ nm}$

$n_i = 4$, $n_f = 2$, $\lambda = 2.75 \text{ nm}$

$n_i = 4$, $n_f = 3$, $\lambda = 4.70 \text{ nm}$

$n_i = 3$, $n_f = 1$, $\lambda = 4.12 \text{ nm}$

$n_i = 3$, $n_f = 2$, $\lambda = 6.59 \text{ nm}$

$n_i = 2$, $n_f = 1$, $\lambda = 10.9 \text{ nm}$

6-12 $\Delta E = \frac{hc}{\lambda} = \left(\frac{\hbar^2}{8mL^2}\right)[2^2 - 1^2]$ and $L = \left[\frac{(3/8)\hbar\lambda}{mc}\right]^{1/2} = 7.93 \times 10^{-10} \text{ m} = 7.93 \text{ \AA}$.

6-13 (a) Proton in a box of width $L = 0.200 \text{ nm} = 2 \times 10^{-10} \text{ m}$

$$E_1 = \frac{\hbar^2}{8m_p L^2} = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(1.67 \times 10^{-27} \text{ kg})(2 \times 10^{-10} \text{ m})^2} = 8.22 \times 10^{-22} \text{ J}$$

$$= \frac{8.22 \times 10^{-22} \text{ J}}{1.60 \times 10^{-19} \text{ J/eV}} = 5.13 \times 10^{-3} \text{ eV}$$

(b) Electron in the same box:

$$E_1 = \frac{h^2}{8m_e L^2} = \frac{(6.626 \times 10^{-34} \text{ J}\cdot\text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(2 \times 10^{-10} \text{ m})^2} = 1.506 \times 10^{-18} \text{ J} = 9.40 \text{ eV} .$$

(c) The electron has a much higher energy because it is much less massive.

6-16 (a) $\psi(x) = A \sin\left(\frac{\pi x}{L}\right)$, $L = 3 \text{ \AA}$. Normalization requires

$$1 = \int_0^L |\psi|^2 dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{LA^2}{2}$$

$$\text{so } A = \left(\frac{2}{L}\right)^{1/2}$$

$$P = \int_0^{L/3} |\psi|^2 dx = \left(\frac{2}{L}\right)^{1/2} \int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{\pi} \int_0^{\pi/3} \sin^2 \phi d\phi = \frac{2}{\pi} \left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8} \right] = 0.1955 .$$

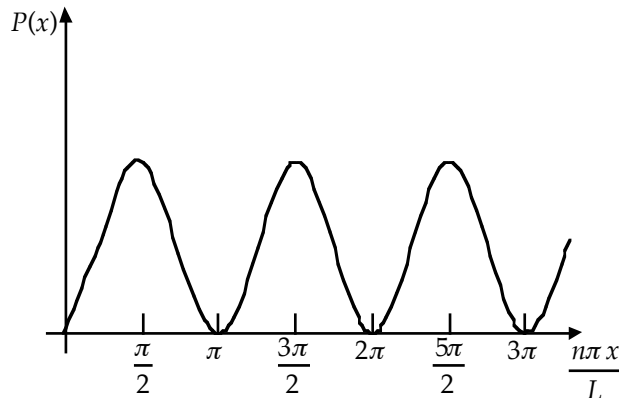
(b) $\psi = A \sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \frac{2}{L} \int_0^{L/3} \sin^2\left(\frac{100\pi x}{L}\right) dx = \frac{2}{L} \left(\frac{L}{100\pi}\right) \int_0^{100\pi/3} \sin^2 \phi d\phi = \frac{1}{50\pi} \left[\frac{100\pi}{6} - \frac{1}{4} \sin\left(\frac{200\pi}{3}\right) \right]$$

$$= \frac{1}{3} - \left[\frac{1}{200\pi} \right] \sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.3319$$

(c) Yes: For large quantum numbers the probability approaches $\frac{1}{3}$.

6-18 Since the wavefunction for a particle in a one-dimension box of width L is given by $\psi_n = A \sin\left(\frac{n\pi x}{L}\right)$ it follows that the probability density is $P(x) = |\psi_n|^2 = A^2 \sin^2\left(\frac{n\pi x}{L}\right)$, which is sketched below:



From this sketch we see that $P(x)$ is a *maximum* when $\frac{n\pi x}{L} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots = \pi\left(m + \frac{1}{2}\right)$ or when

$$x = \frac{L}{n}\left(m + \frac{1}{2}\right) \quad m = 0, 1, 2, 3, \dots, n.$$

Likewise, $P(x)$ is a *minimum* when $\frac{n\pi x}{L} = 0, \pi, 2\pi, 3\pi, \dots = m\pi$ or when

$$x = \frac{Lm}{n} \quad m = 0, 1, 2, 3, \dots, n$$

6-29 (a) Normalization requires

$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = C^2 \int_0^{\infty} e^{-2x} (1 - e^{-x})^2 dx = C^2 \int_0^{\infty} (e^{-2x} - 2e^{-3x} + e^{-4x}) dx$. The integrals are elementary and give $1 = C^2 \left\{ \frac{1}{2} - 2\left(\frac{1}{3}\right) + \frac{1}{4} \right\} = \frac{C^2}{12}$. The proper units for C are those of $(\text{length})^{-1/2}$ thus, normalization requires $C = (12)^{1/2} \text{ nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where ψ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0 = \frac{d\psi}{dx} = C \{-e^{-x} + 2e^{-2x}\} = Ce^{-x} \{2e^{-x} - 1\}.$$

The RHS vanishes when $x = \infty$ (a minimum), and when $2e^{-x} = 1$, or $x = \ln 2 \text{ nm}$. Thus, the most likely position is at $x_p = \ln 2 \text{ nm} = 0.693 \text{ nm}$.

(c) The average position is calculated from

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = C^2 \int_0^{\infty} x e^{-2x} (1 - e^{-x})^2 dx = C^2 \int_0^{\infty} x (e^{-2x} - 2e^{-3x} + e^{-4x}) dx.$$

The integrals are readily evaluated with the help of the formula $\int_0^{\infty} x e^{-ax} dx = \frac{1}{a^2}$ to

get $\langle x \rangle = C^2 \left\{ \frac{1}{4} - 2\left(\frac{1}{9}\right) + \frac{1}{16} \right\} = C^2 \left\{ \frac{13}{144} \right\}$. Substituting $C^2 = 12 \text{ nm}^{-1}$ gives

$$\langle x \rangle = \frac{13}{12} \text{ nm} = 1.083 \text{ nm}.$$

We see that $\langle x \rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of x larger than x_p are weighted more heavily in the calculation of the average.

6-30 The possible particle positions within the box are weighted according to the probability density $|\psi|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right)$. The position is calculated as

$$\langle x \rangle = \int_0^L x |\psi|^2 dx = \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx. \text{ Making the change of variable } \theta = \frac{n\pi x}{L} \text{ (so that}$$

$$d\theta = \frac{\pi dx}{L}) \text{ gives } \langle x \rangle = \frac{2L}{\pi^2} \int_0^\pi \theta \sin^2 n\theta d\theta. \text{ Using the trigonometric identity}$$

$$2 \sin^2 \theta = 1 - \cos 2\theta, \text{ we get } \langle x \rangle = \frac{L}{\pi^2} \left\{ \int_0^\pi \theta d\theta - \int_0^\pi \theta \cos 2n\theta d\theta \right\}. \text{ An integration by parts}$$

shows that the second integral vanishes, while the first integrates to $\frac{\pi^2}{2}$. Thus, $\langle x \rangle = \frac{L}{2}$,

independent of n . For the computation of $\langle x^2 \rangle$, there is an extra factor of x in the

integrand. After changing variables to $\theta = \frac{\pi x}{L}$ we get

$$\langle x^2 \rangle = \frac{L^2}{\pi^3} \left\{ \int_0^\pi \theta^2 d\theta - \int_0^\pi \theta^2 \cos 2n\theta d\theta \right\}. \text{ The first integral evaluates to } \frac{\pi^3}{3}, \text{ the second may}$$

be integrated by parts twice to get

$$\int_0^\pi \theta^2 \cos 2n\theta d\theta = -\frac{1}{n} \int_0^\pi \theta \sin 2n\theta d\theta = \left(\frac{1}{2n^2} \right) \theta \cos 2n\theta \Big|_0^\pi = \frac{\pi}{2n^2}.$$

$$\text{Then } \langle x^2 \rangle = \frac{L^2}{\pi^3} \left\{ \frac{\pi^3}{3} - \frac{\pi}{2n^2} \right\} = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}.$$

6-31 The symmetry of $|\psi(x)|^2$ about $x=0$ can be exploited effectively in the calculation of average values. To find $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx$$

We notice that the integrand is antisymmetric about $x=0$ due to the extra factor of x (an odd function). Thus, the contribution from the two half-axes $x > 0$ and $x < 0$ cancel exactly, leaving $\langle x \rangle = 0$. For the calculation of $\langle x^2 \rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

$$\langle x^2 \rangle = \int_0^\infty x^2 |\psi|^2 dx = 2C^2 \int_0^\infty x^2 e^{-2x/x_0} dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting

$$\text{for } C^2, \text{ we get } \langle x^2 \rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2} \text{ and } \Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}. \text{ In}$$

calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x} |\psi|^2 dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_0} dx = -2C^2 \left(\frac{x_0}{2} \right) e^{-2x/x_0} \Big|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of x_0 .