Recap: Schwarzschild spacetimes

Falling into and crossing Schwarzschild horizon

Let's look at timelike geodesics

$$\frac{dt}{d\tau} = \frac{E}{1 - \frac{2GM/c^2}{r}} \; ; \; \frac{dr}{c\,d\tau} = -\sqrt{E^2 - 1 + \frac{2GM/c^2}{r}}$$

$$\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = E\left(1 - \frac{2GM/c^2}{r}\right)^{-1}(-1)\left(E^2 - 1 + \frac{2GM/c^2}{r}\right)^{-1/2}$$

Falling in from rest at $r = \infty$, $\Rightarrow E = 1$ $\left(E = \frac{e}{mc^2} = 1\right)$

As a result: $\frac{dt}{dr} = -\left(1 - \frac{r_s}{r}\right)^{-1} \frac{r^{1/2}}{r_s^{1/2}}$ $\quad (r_s \equiv 2GM/c^2)$

Define new variable: $\epsilon \equiv r - r_s$; $d\epsilon = dr$

Therefore: $1 - \frac{r_s}{r} = \frac{r - r_s}{r} = \frac{\epsilon}{r}$;

Thus, $dt = -\frac{d\epsilon}{\frac{\epsilon}{r}\left(\frac{r_s}{r}\right)^{1/2}} = -\frac{r^{3/2}\, d\epsilon}{r_s^{1/2}\, \epsilon}$

or

$$\boxed{dt = \frac{(\epsilon + r_s)^{3/2}}{r_s^{1/2}} \frac{d\epsilon}{\epsilon}}$$

Limit: $\epsilon \to 0$ (Rocket approaches r_s from $r > r_s$).

Integral diverges as $\ln \epsilon$.

So timelike particles take infinite coordinate time to reach $r = r_s$ surface

But, we previously worked out the particle reaches r_s in finite proper time, τ.
Since $r = r_s$ is not a physical singularity, t must be a bad coordinate.

Inside $r = r_s$

Consider metric at $r < r_s$

Redefine $\boxed{\epsilon = r_s - r}$ (rather than $\epsilon = r - r_s$)

$$-ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$= \left(\frac{r - r_s}{r}\right)(c\,dt)^2 - \left(\frac{r - r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$-ds^2 = -\frac{\epsilon}{r}(c\,dt)^2 - \left(\frac{-\epsilon}{r}\right)^{-1} d\epsilon^2 - r^2 d\Omega^2$$

$$ds^2 = \frac{\epsilon}{r_s - \epsilon}(c\,dt)^2 - \frac{r_s - \epsilon}{\epsilon} d\epsilon^2 + r^2 d\Omega^2$$

- Consider curves for which $dt = d\theta = d\phi = 0$ ("spacelike" usually)

$$ds^2 = -\frac{(r_s - \epsilon)}{\epsilon} d\epsilon^2$$

Since $ds^2 < 0$, both ϵ (and r) become timelike at $r < r_s$

- Consider curves for which $d\epsilon = d\theta = d\phi = 0$

$$ds^2 = \frac{\epsilon}{r_s - \epsilon} c^2 dt^2$$

Because $ds^2 > 0$, dt is spacelike at $r < r_s$

Since infalling object must follow timelike worldline, ϵ must increase (time marches on!).

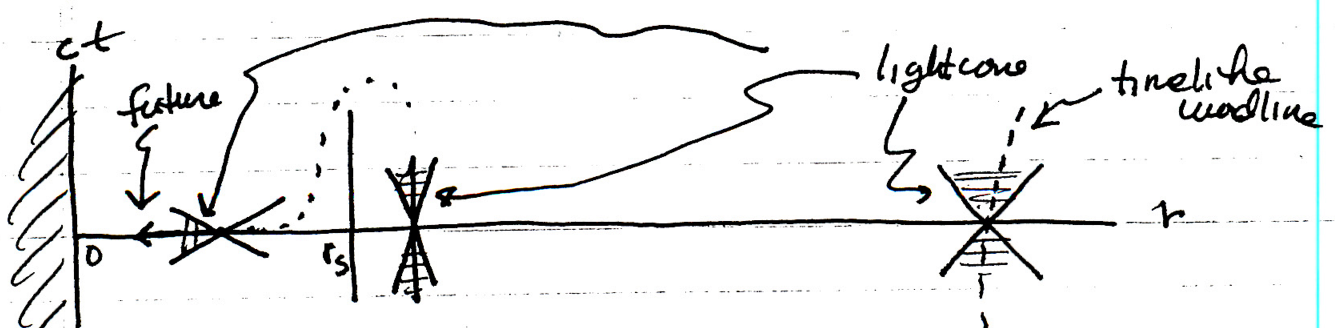
But $\epsilon = r_s - r$

Therefore if ϵ increases, r must decrease. Result is that object inevitably reaches $r=0$.

- Light signals: what happens if someone at $r < r_s$ tries to send out photon to $r > r_s$. Since ϵ must increase, r again must decrease, no matter how photon is initially directed. Photon cannot escape!

Nothing inside $r = r_s$ can emerge from within r_s. Future for everyone (timelike + lightlike) is curvature singularity at $r=0$. This is why surface at $r = r_s$ is called the Event Horizon

On Schwarzschild coordinates



Horizons:

Once particle crosses $r=r_s$, it cannot be "seen" by external observers. To be "seen" means sending out a photon that reaches external observer

Again: why $r = r_s$ is the event horizon

Closing off of light cones

Radial geodesics: $-ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2$

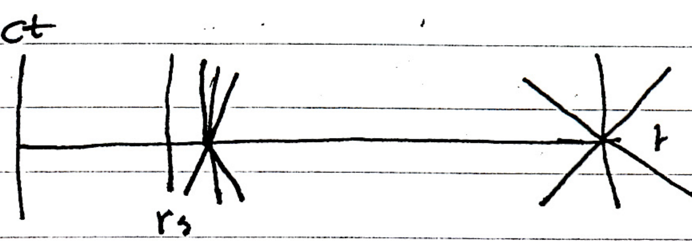
~~Light~~ Light like: $ds^2 = 0$. Implying

$$\left(1 - \frac{r_s}{r}\right) c^2 dt^2 = \frac{dr^2}{1 - \frac{r_s}{r}}$$

$$\Rightarrow \left(\frac{c\,dt}{dr}\right)^2 = \frac{1}{\left(1 - \frac{r_s}{r}\right)^2}$$

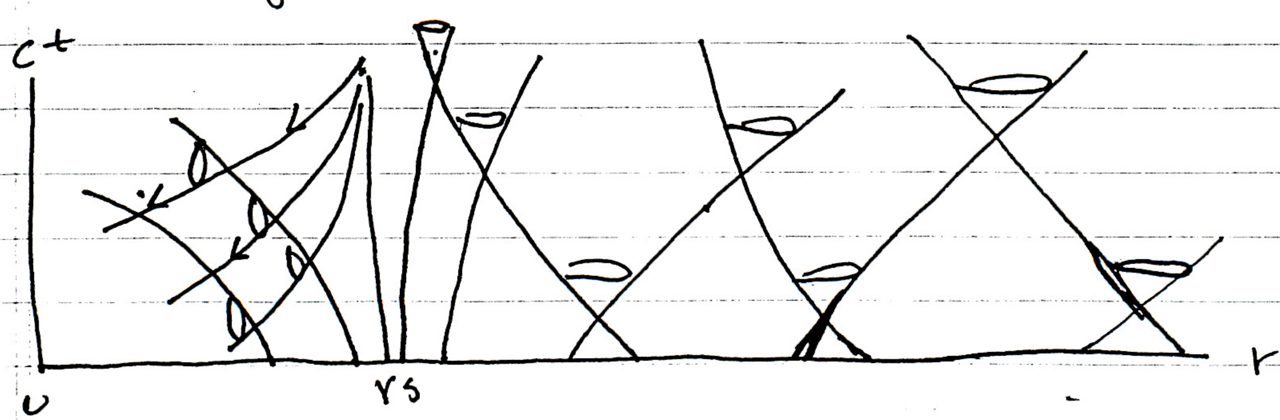
$$\boxed{\frac{c\,dt}{dr} = \pm \frac{1}{1 - \frac{r_s}{r}}}$$

Limit $r \gg r_s$: $\frac{c\,dt}{dr} = \pm 1 \; (45°)$



Limit $r \to r_s$: $\frac{c\,dt}{dr} \to \pm\infty \; (90°)$

So as $r \to r_s$, light moves only along time axis and thus doesn't get beyond $r = r_s$



Let's integrate lightlike geodesics

Outgoing: $$c\,dt = + \frac{dr}{1 - r_s/r} = \frac{dr\,(r/r_s)}{(r/r_s) - 1}$$

$$\therefore\; c\,dt = \frac{r_s\, d(r/r_s)\,(r/r_s)}{r/r_s - 1}$$

$$ct = r_s \int \frac{dx \cdot x}{x - 1} \quad :$$

Let $u = x - 1$, $du = dx$ $\Rightarrow$ $$ct = r_s \int \frac{du\,(u+1)}{u}$$

or $$ct = r_s \left[\int du + \int \frac{du}{u} \right] = r_s \left[u + \ln u \right] + \text{const.}$$

$$ct = r_s \left[\frac{r}{r_s} - 1 + \ln\left(\frac{r}{r_s} - 1\right) \right] + \text{const.}$$

$$ct = r + r_s \ln\left(\frac{r}{r_s} - 1\right) + \text{const}_1$$

$$\boxed{\text{or } ct = r + r_s \ln\left|\frac{r}{r_s} - 1\right| + \text{const}_1 \qquad (\text{outgoing})}$$

$$\boxed{ct = -r - r_s \ln\left|\frac{r}{r_s} - 1\right| + \text{const}_2 \qquad (\text{ingoing})}$$

Eddington Finkelstein Coordinates

To describe infall, or more generally gravitational collapse, find coordinates in which metric does not diverge at $r = r_s$. Recall const, remains constant along entire worldline of ingoing light like geodesic. Let's define use new coordinate $v = \text{const}_2$. This will be a good coordinate even when light ray passes through $r = r_s$. Therefore changing v means going from 1 light ray to another.

$$\therefore \boxed{ct = -r - r_s \ln\left[\frac{r}{r_s} - 1\right] + v}$$

(Ingoing: as t increases, r decreases.)

$$\text{or } ct = v - r - r_s \ln|r - r_s| + r_s \ln r_s$$

$$\therefore c\,dt = dv - dr - \frac{r_s\,dr}{r - r_s} \qquad (\text{assume } r > r_s)$$

$$d(ct) = dv - dr\left(1 + \frac{r_s}{r - r_s}\right)$$

$$d(ct) = dv - dr\left(\frac{r}{r - r_s}\right)$$

$$\boxed{d(ct) = dv - \frac{dr}{1 - \frac{r_s}{r}}} \qquad (1)$$

Schwarzschild Metric

$$-ds^2 = \left(1 - \frac{r_s}{r}\right) d(ct)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$\therefore -ds^2 = \left(1 - \frac{r_s}{r}\right)\left[dv - \frac{dr}{1 - \frac{r_s}{r}}\right]^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\Omega^2$$

$$-ds^2 = \left(1 - \frac{r_s}{r}\right)\left[dv^2 - \frac{2\,dv\,dr}{1 - \frac{r_s}{r}} + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)^2}\right] - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\Omega^2$$

$$-ds^2 = \left(1-\frac{r_s}{r}\right) dv^2 - 2\,dv\,dr + \cancel{\frac{dr^2}{1-\frac{r_s}{r}}} - \cancel{\frac{dr^2}{1-\frac{r_s}{r}}} - r^2 d\Omega^2$$

$$\therefore \boxed{-ds^2 = \left(1-\frac{r_s}{r}\right) dv^2 - 2\,dv\,dr - r^2 d\Omega^2} \quad (2)$$

(Note same expression for $r < r_s$)

<u>Advantages of new coordinates</u>

- No coordinate singularity at $r = r_s$
- Same expression for ds^2 is valid in range $0 \le r \le \infty$
- <u>Not</u> new geometry. Same geometry as Schwarzschild, but different coordinate labels for spacetime points

<u>Look at radial lightlike geodesics</u>

$-ds^2 = 0$; $d\Omega^2 = 0$ (divide $-ds^2/dr^2 = \ldots$)

$$\left(1-\frac{r_s}{r}\right)\left(\frac{dv}{dr}\right)^2 - 2\frac{dv}{dr} = 0$$

$$\frac{dv}{dr}\left[\left(1-\frac{r_s}{r}\right)\left(\frac{dv}{dr}\right) - 2\right] = 0$$

<u>Two solutions</u>

(i) $\frac{dv}{dr} = 0 \Rightarrow v = \text{const.}$ (incoming ray)

$$\frac{dv}{dr} = \frac{2}{1-\frac{r_s}{r}} \Rightarrow v = 2r + 2r_s \ln\left|\frac{r}{r_s} - 1\right| + \text{const}$$

(outgoing)

<u>Define</u>:

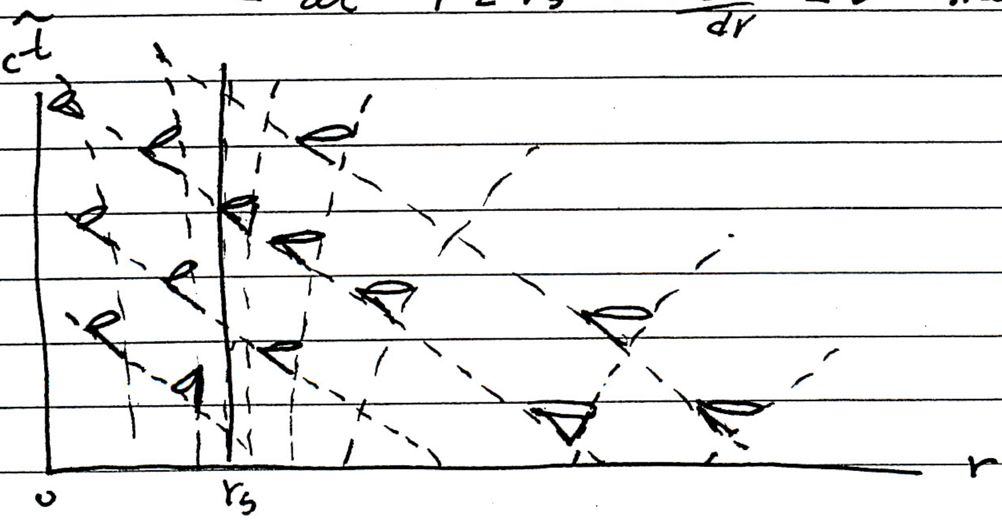
$$c\tilde{t} = v - r$$

$$\boxed{\begin{array}{l} \text{Incoming: } c\tilde{t} = \text{const} - r \text{; since } v = \text{const.} \\ \text{outgoing: } c\tilde{t} = \underset{\uparrow \text{(not } 2r)}{r} + 2r_s \ln\left|\frac{r}{r_s} - 1\right| + \text{const}' \end{array}}$$

In (A) $\dfrac{c\,d\tilde{t}}{dr} = -1$: $-45°$ angle with respect to r: at all r

Out (B) $c\dfrac{d\tilde{t}}{dr} = \dfrac{dv}{dr} - 1 = \dfrac{2}{1 - \frac{r_s}{r}} - 1$

- $\dfrac{c\,d\tilde{t}}{dr} = \infty$ at $r = r_s$, but $\dfrac{c\,d\tilde{t}}{dr} \to 1$ at $r \gg r_s$
  - at $r > r_s$; $\dfrac{c\,d\tilde{t}}{dr} > 0$: positive angle
  - at $r < r_s$: $\dfrac{c\,d\tilde{t}}{dr} < 0$: negative angle



- No flip of space → time, time → space at $r = r_s$
- After crossing horizon future is directed toward $r = 0$ singularity
- light emitted at $r < r_s$ never reaches observer at $r > r_s$

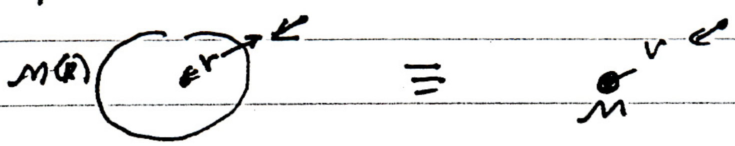
Collapse to a BH

Suppose spherical mass with $M > M_{crit}$ starts to collapse $\begin{Bmatrix} M_{crit} = M_{ch} = 1.4\ M_\odot \text{ for white Dwarf} \\ M_{crit} \simeq 3 M_\odot \quad \text{for neutron star} \end{Bmatrix}$

Assumptions:

(1) Internal pressure drops as reactions become endothermic (Fe cores) - So let pressure vanish

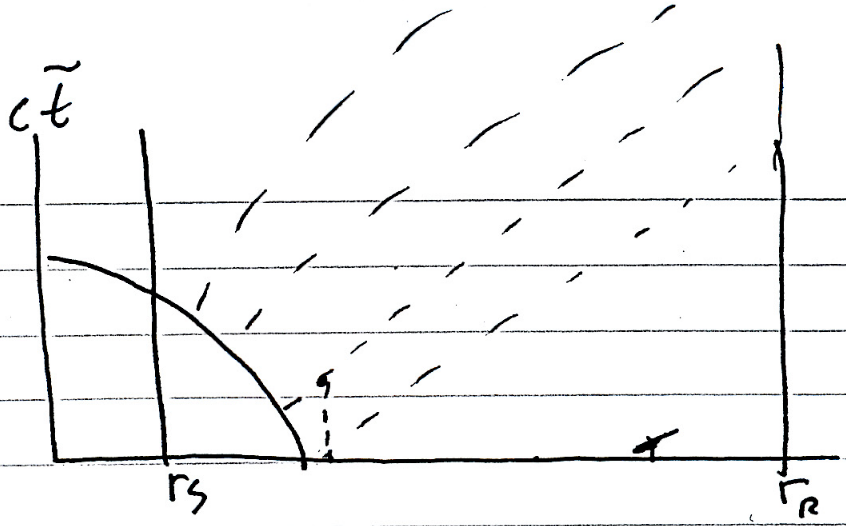
(2) Outer surface of star follows radial, infalling, time like geodesic. Recall in Newtonian theory gravitational field outside spherical mass has same force and ~~of~~ potential as point at center of sphere with same mass



Birkhoff's theorem is analogue in GR. Can assume Schwarzschild spacetime at surface of sphere.

(3) Consider 2 observers at rest at $r \gg r_s$.

- One free falls on surface of star.
- The other remains fixed at r_R



Signals detected at (r_R, ct_R) but emitted at (r_E, ct_E).

Recall outgoing light rays:

$$v = 2r + 2r_S \ln\left(\frac{r}{r_S} - 1\right) + \text{const}.$$

$\hookrightarrow$

Kruskal - Szekeres Coordinates

alternative to E.F. coordinates

$(t, r) \to (u, v)$

(1a) $u = \left(\frac{r}{2GM/c^2} - 1\right)^{1/2} e^{r/\frac{4GM}{c^2}} \cosh\left(\frac{ct}{4GM/c^2}\right)$

$r > 2GM/c^2$

(1b) $v = \left(\frac{r}{2GM/c^2} - 1\right)^{1/2} e^{r/\frac{4GM}{c^2}} \sinh\left(\frac{ct}{4GM/c^2}\right)$

(1c) $u = \left(1 - \frac{r}{2GM/c^2}\right)^{1/2} e^{r/\frac{4GM}{c^2}} \sinh\left(\frac{ct}{4GM/c^2}\right)$

$r < 2GM/c^2$

(1d) $v = \left(1 - \frac{r}{2GM/c^2}\right)^{1/2} e^{r/\frac{4GM}{c^2}} \cosh\left(\frac{ct}{4GM/c^2}\right)$

Metric in these coordinates:

$$\boxed{ds^2 = -\frac{32(GM/c^2)^3}{r} e^{-r/\frac{2GM}{c^2}} (dv^2 - du^2) + r^2 d\Omega^2} \quad (2)$$

v timelike, u spacelike always

Now r not a coordinate, rather $r = r(u, v)$, given by:

$$u^2 - v^2 = \left(\frac{r}{2GM/c^2} - 1\right) e^{\frac{r}{\frac{2GM}{c^2}}} \underbrace{\left[\cosh^2 - \sinh^2\right]}_{1}$$

$$(3) \quad \boxed{u^2 - v^2 = \left(\frac{r}{2GM/c^2} - 1\right) e^{r/\frac{2GM}{c^2}}}$$

Good for $r \gtrless 2GM/c^2$

Comments

(1) Metric not singular at $r = 2GM/c^2$
but is singular at $r = 0$ where it should be!

(2) Radial light like geodesics:
$ds^2 = d\Omega^2 = 0$. From eq. (2) $dv^2 = du^2$
$dv = \pm du$

- Light cone doesn't close up at $r = 2GM/c^2$

Recap: Schwarzschild spacetime in Eddington–Finkelstein coordinates

- Transform $(t, r) \to (v, r)$

$$\boxed{ct = -r - r_s \ln\left|\frac{r}{r_s} - 1\right| + v} \quad (1)$$

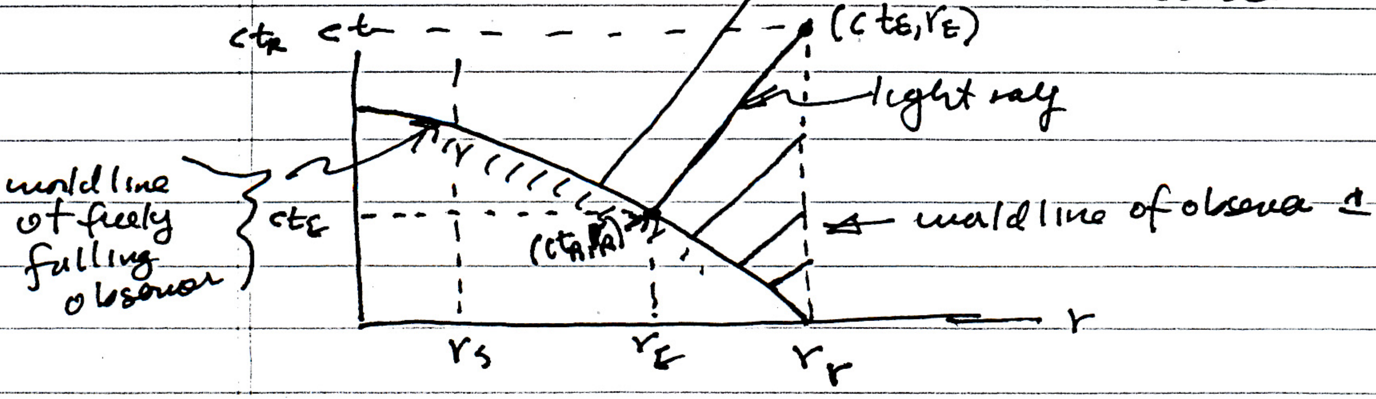
- Metric: $$\boxed{-ds^2 = \left(1 - \frac{r_s}{r}\right) dv^2 - 2\,dv\,dr - r^2 d\Omega^2} \quad (2)$$

- lightlike geodesics:

$v = \text{const}$: ingoing

$$\boxed{v = 2\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const}'} \text{ : outgoing} \quad (3)$$

- Gravitational collapse:
  - 2 observers initially at $r = r_R$ ($\gg r_s$)
  - One stops at r_R, while the other free falls into BH: Same as surface of star
  - what does observer at r_R observe



along outgoing light ray, eq. (3) implies

$$v - 2\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) = \text{const. along lightlike geodesic}$$

observer who detects it at (ct_R, r_R)

same const for emission & detection events

$$v_E - 2\left(r_E + r_S \ln\left|\frac{r_E}{r_S} - 1\right|\right) = v_R - 2\left(r_R + r_S \ln\left|\frac{r_R}{r_S} - 1\right|\right)$$

- $r_E \sim r_S$: ln term dominates LHS
- $r_R \gg r_S$: $v_R - 2r_R$ term dominates RHS

$$\therefore\ -2 r_S \ln\left(\frac{r_E}{r_S} - 1\right) \simeq v_R - 2r_R$$

But at $r \gg r_S$: $ct_R \simeq v_R - r_R$ (from eq. (1))

$$\Rightarrow\ -2 r_S \ln\left(\frac{r_E}{r_S} - 1\right) \simeq ct_R + r_R - 2r_R = ct_R - r_R$$

$$\boxed{\therefore\ \frac{r_E}{r_S} - 1 \simeq \exp\left[-\left(\frac{ct_R - r_R}{2 r_S}\right)\right]} \quad (4)$$

$$r_E = r_S \left\{1 + \exp\left[-\frac{(ct_R - r_R)}{4GM/c^2}\right]\right\}$$

Thus r_E approaches r_S with timescale $4GM/c^3$

sphere rapidly approaches Horizon. ($t_* \sim \frac{4GM}{c^3} = 2 \times 10^{-5}$ s for $M = M_\odot$)

Redshift: In proper time $\Delta\tau_E$, infalling observer moves distance $\boxed{\Delta r_E = |u^r|\,\Delta\tau}$ closer to r_S.

From eq (4) $$\frac{\Delta r_E}{r_S} = \exp\left[\frac{-[ct_R - r_R]}{2r_S}\right]\left[\frac{-c\,\Delta t_R}{2r_S}\right] \quad (r_R = \text{const.})$$

$$\boxed{\Delta r_E = \left(-\frac{c\,\Delta t_R}{2}\right)\exp\left[-\left(\frac{ct_R - r_R}{2r_s}\right)\right]} \quad (5)$$

But for small $\Delta\tau_E$, $\Delta r_E = U^r \Delta\tau_E$

Recall from free-fall solution: $\frac{1}{c}\frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{2GM/c^2}{r}}$

Free fall from rest $\Rightarrow E = 1$,

Therefore: $\frac{dr}{d\tau} = -c\sqrt{\frac{2GM/c^2}{r}}$

or $$\boxed{u^r = \frac{dr}{d\tau} = -c\sqrt{\frac{r_s}{r}}} \quad (6)$$

As a result, combing equations (5) and (6):

$$-c\sqrt{\frac{r_s}{r}}\,\Delta\tau_E = -\frac{c\,\Delta t_R}{2}\exp\left[-\left(\frac{ct_R - r_R}{2r_s}\right)\right]$$

Since $r \sim r_s$, we have

$$\Delta\tau_E \simeq \frac{\Delta t_R}{2}\exp\left[-\left(\frac{ct_R - r_R}{2r_s}\right)\right]$$

If $\Delta\tau$ correspond to periods, $\nu_E \sim 1/\Delta\tau_E$; $\nu_R \sim 1/\Delta t_R$

$$\frac{1}{\nu_E} \simeq \frac{1}{2\nu_R}\exp\left[-\left(\frac{ct_R - r_R}{2r_s}\right)\right]$$

$$\boxed{\nu_R \simeq \frac{\nu_E}{2}\exp\left[-\left(\frac{t_R - r_R/c}{2r_s/c}\right)\right]}$$

Light is redshifted as $t_R > r_R/c$ on rapid ~~the~~ timescale

$t_{crit} \simeq 2r_s/c = 4GM/c^3 = 2\times10^{-5}\,(M/M_\odot)$ sec.

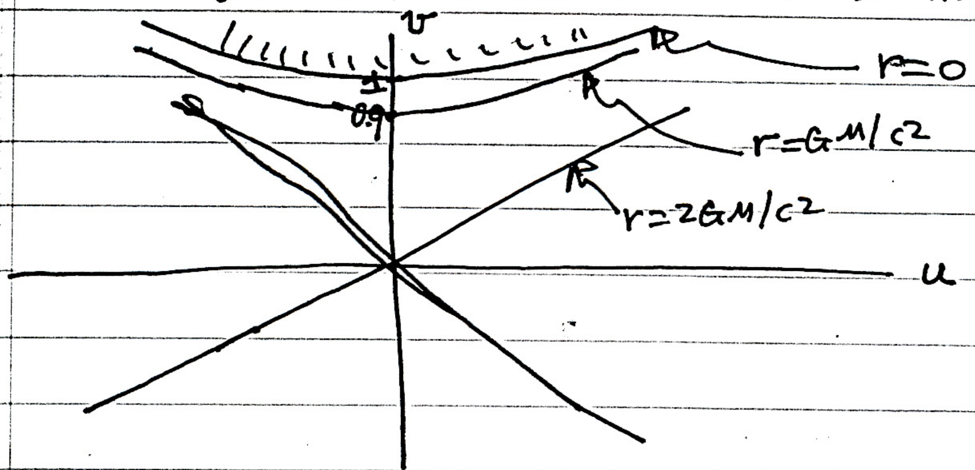
- photon energy decrease means loss of luminosity.
- All evidence of object disappears except for warp in spacetime

Lightlike geodesics are st. lines making 45° angle with u, V axes.

③ Lines of constant r are hyperbolae

$$u^2 - v^2 = \left(\frac{r}{r_s} - 1\right) \exp(r/r_s) = \text{const}$$

- $r > r_s$: const > 0
- $r < r_s$: const < 0
- $r = r_s$: const $= 0$ (45° st. lines)



(a) at $r = r_s = 2GM/c^2$: $u^2 - v^2 = 0$: correspond to lightlike geodesics propagating from $u, v = (0,0)$ (light ray at horizon is "frozen" at $r = 2GM/c^2$

(b) at $r = 0$

$$u^2 - v^2 = -1 \quad \text{or} \quad v^2 = u^2 + 1$$

$$v = \pm\sqrt{u^2+1} \quad : \text{pick } v = +\sqrt{u^2+1},$$

since $v > 0$ at $r < r_s$ (eq. 1d)

Singularity an hyperbola in SK coordinates

(c) at $r < 2GM/c^2$: say $r = GM/c^2$

$$v^2 = u^2 - \left(\frac{r}{r_s} - 1\right) e^{r/r_s} = u^2 + \left(1 - \frac{r}{r_s}\right) e^{r/r_s}$$

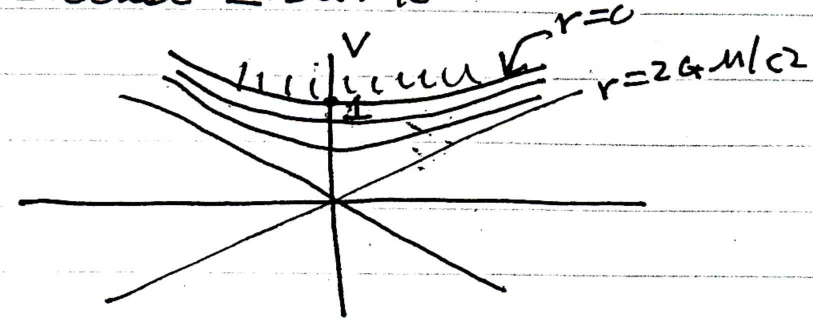
$$v = +\sqrt{u^2 + \left(1 - \frac{r}{r_s}\right) e^{r/r_s}}$$

In this case

$$v = +\sqrt{u^2 + 1/2\, e^{1/2}} = \sqrt{u^2 + .82}$$

$$u = 0 \Rightarrow v = 0.9$$

We can compute sequence of "horizontal" hyperbolae along which $r = \text{const} < 2GM/c^2$



(4) What do ~~[illegible]~~ curves of $r = \text{const}$ look like when $r > 2GM/c^2$. From eq. (3) we have:

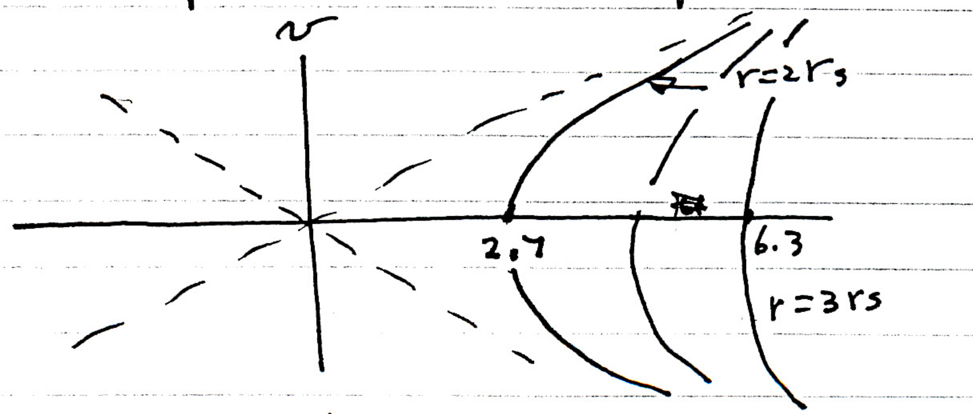
$$u^2 - v^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s}$$

$$v = +\sqrt{u^2 - \left(\frac{r}{r_s} - 1\right) e^{r/r_s}}$$

{ + sign from 2-b at $r > r_s$

Try $r = 2r_s = 4GM/c^2$

$$v(r = 2r_s) = \sqrt{u^2 - (2-1)e^2} = \sqrt{u^2 - 7.39}$$



So at $r > r_s$, surfaces of const. r are "vertical" hyperbolae.

(5) Lines of constant t

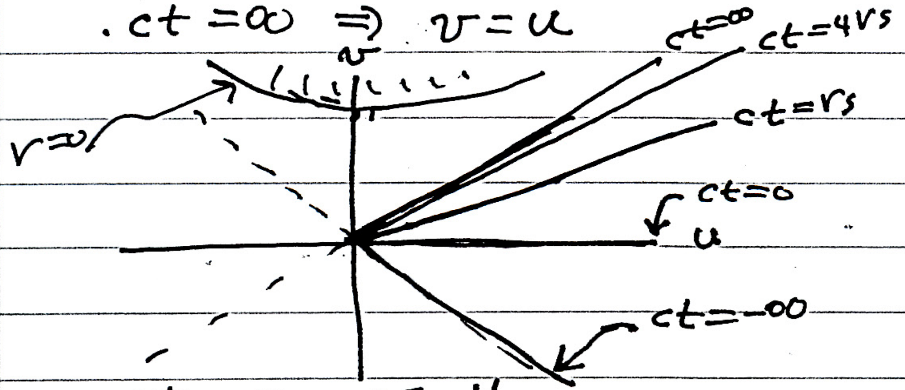
From eqs. (1)

$$\tanh\left(\frac{ct}{2r_s}\right) = \begin{cases} \dfrac{v}{u} & : r > r_s \\ \dfrac{u}{v} & : r < r_s \end{cases}$$

Therefore fixed values of t correspond to lines of constant v/u; i.e., st. lines through the origin with slope depending on $ct/2r_s$

$r > r_s$ $v = u \cdot \tanh(ct/2r_s) = u \left\{ \dfrac{e^{\left(\frac{ct}{2r_s}\right)} - e^{-\left(\frac{ct}{2r_s}\right)}}{e^{\left(\frac{ct}{2r_s}\right)} + e^{-\left(\frac{ct}{2r_s}\right)}} \right\}$

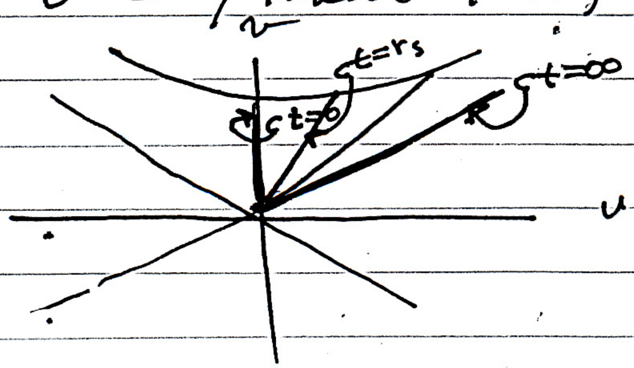
- $ct = 0 \Rightarrow v = 0$
- $ct = r_s \Rightarrow v = u \cdot \left[\dfrac{e^{1/2} - e^{-1/2}}{e^{1/2} + e^{-1/2}}\right] = 0.46u$
- $ct = 4r_s \Rightarrow v = 0.96\,u$
- $ct = \infty \Rightarrow v = u$



slope of st. lines increase with increasing t

$ct = -\infty$ $v = -u$

$r < r_s$ $v = u/\tanh(ct/2r_s)$ or $u = v \cdot \tanh(ct/2r_s)$



slope of st lines increase with decreasing t

remember t is spacelike at $r < r_s$

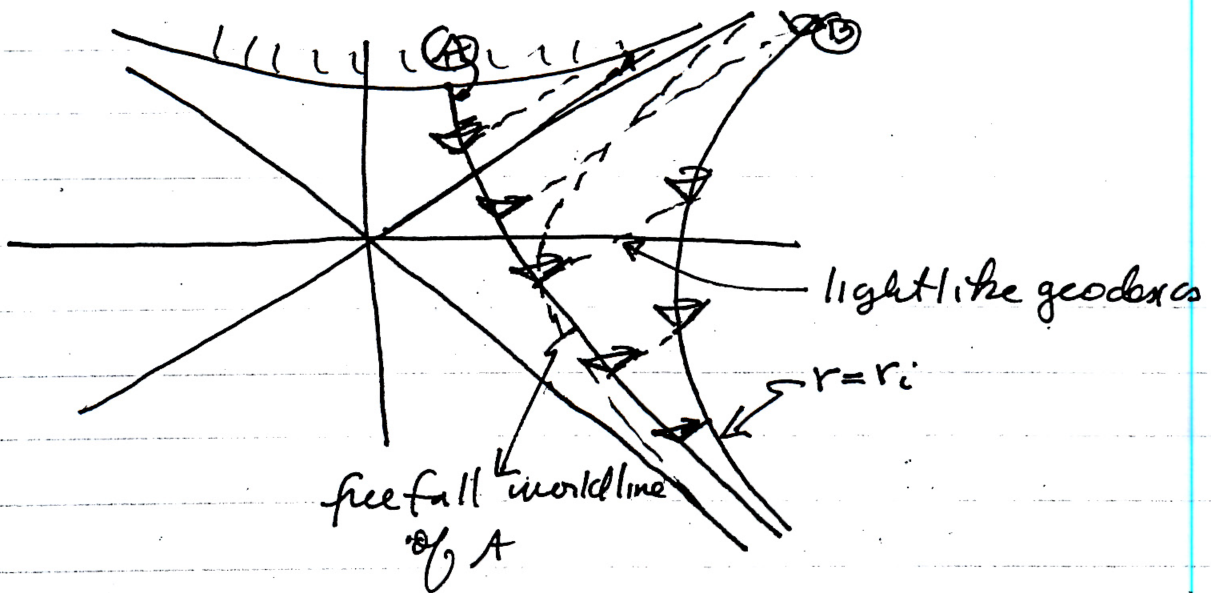
$ct = 0 \Rightarrow u = 0$

$ct = r_s \Rightarrow u = .46\,v$ or $v = 2.17u$ ($\theta = 73°$)

World lines

- For entire range of (u, v, θ, ϕ)
  $g_{vv} < 0 \Rightarrow$ direction of v is timelike
  $g_{uu} > 0 \Rightarrow$ direction of u is spacelike

- Two observers at initial distance $r = r_i \gg r_s$ and at $t \to -\infty$.
  Observer A starts to free fall
  Observer B stays fixed at r_i : A signals B

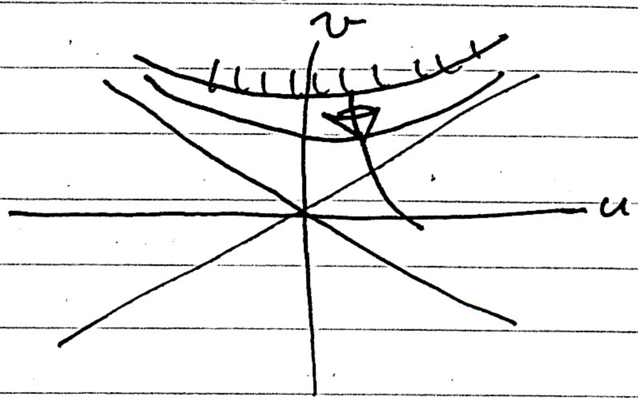


Comments

(i) Emission pulses spaced at constant $\Delta\tau_E$ are received at increasingly large time intervals Δt_R

(ii) Observer at $r = r_i$ ($\gg r_s$) can remain at $r = r_i$ since world line $r = r_i$ is within forward light cone and thus is timelike

(iii) But at $r < r_s$, the curves $r = $ const do not lie within forward light cones. Thus here no timelike observer cannot remain at const r. Rather observer must



move off r and head toward singularity at $r=0$

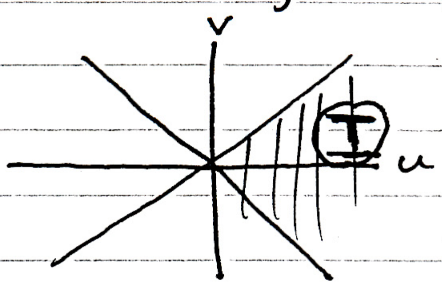
(iv) Kruskal diagram shows directly how distant observer sees observer (A) cross horizon at $t = \infty$

(v) At $r < r_s$, all signals wind up at singularity at $r = 0$.

Kruskal Extension of Schwarzschild Spacetime

What about other regions of u, v plane? Physically, only regions outside world line of collapsing star are relevant for collapse:

$-\infty \le t \le +\infty$; $r_s \le r \le \infty$



This is region (I)

- $-u < v < +u$
- $u > 0$ (eq. 1a)

Extension. Mathematically it is possible to extend things to include entire u,v plane bounded by singularities r=0.

Reconsider r=0

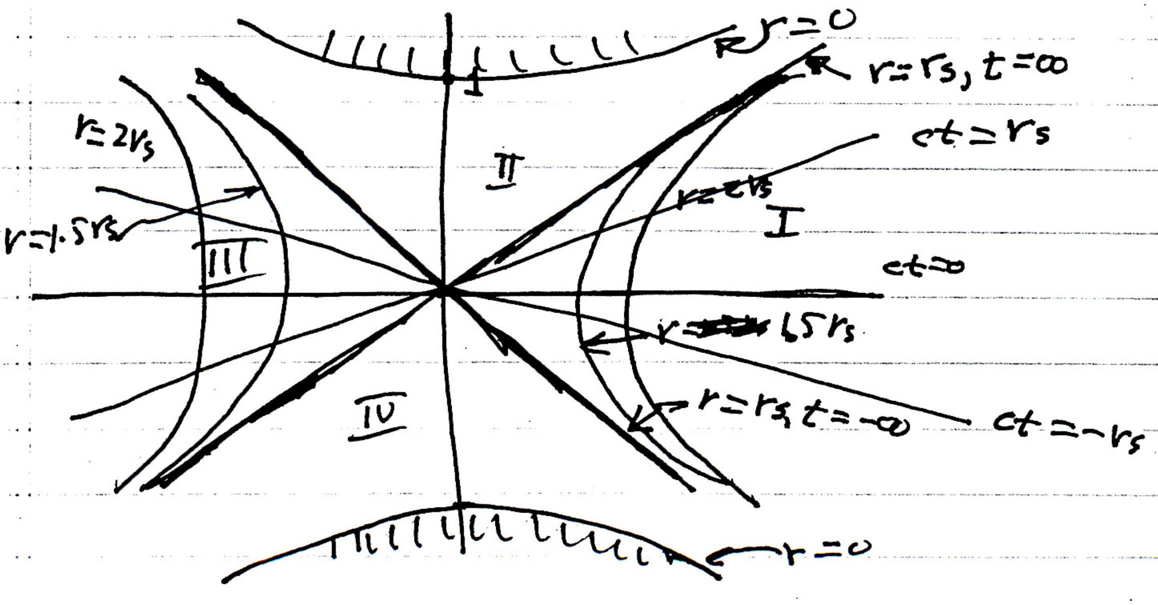
Since $u^2 - v^2 = (\frac{r}{r_s} - 1) e^{r/r_s}$,
r=0 corresponds to

$$u^2 - v^2 = -1$$

or $v^2 = u^2 + 1$

$$v = \pm\sqrt{u^2+1}$$

Previously we restricted r=0 surface to $v = +\sqrt{u^2+1}$
But now consider $v = -\sqrt{u^2+1}$



- Regions I and II cover familiar collapse through event horizon to BH

- Region I: Spacetime outside Schwarzschild BH
  Region II: " interior to BH event horizon

- Particle traveling from I to II can never return, and inevitably hits singularity (hyperbola $r=0$)

- Regions III and IV are inaccessible from regions I and II.

- Region IV similar to II, but in reverse: It is part of spacetime from which particle can escape (into I and III) but not enter

- Important: In region IV singularity $r=0$ is in the past not the future. This is a "white hole" from which particles can emanate!

- Wormhole: The regions I and III are connected by a wormhole at the $u,v=(0,0)$ wormhole; but no particle can really travel between I and III

- What's going on?
  - Kruskal coordinates probe all of Schwarzschild geometry, which consists of a
    - black hole in the future
    - white hole in the past
  - Two "universes" connected at their horizons by a "wormhole"

• Comment

~~Ex~~ Kruskal extension of Schwarzschild spacetime is allowed by classical GR; i.e., GR allows existence of white holes; i.e., particles, photons could spring out of white hole.

But you cannot fall into a white hole since it only exists in the past.

• Can white holes really exist?.

Answer: we don't know. Of course classical GR must break down at singularities; i.e where quantum effects ~ relativistic

• Planck mass:

$$\lambda_{compton} \sim r_s$$

$$\frac{\hbar}{mc} \sim \frac{GM}{c^2}$$

$$\Rightarrow M^2 \simeq \frac{c\hbar}{G}$$

$$M_p = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-15}\, g$$

$$\lambda_p = \frac{\hbar}{c m_p} = \frac{\hbar}{c}\left(\frac{G}{\hbar c}\right)^{1/2} = \left(\frac{\hbar^2}{c^2}\cdot\frac{G}{\hbar c}\right)^{1/2} = \left(\frac{\hbar G}{c^3}\right)^{1/2}$$

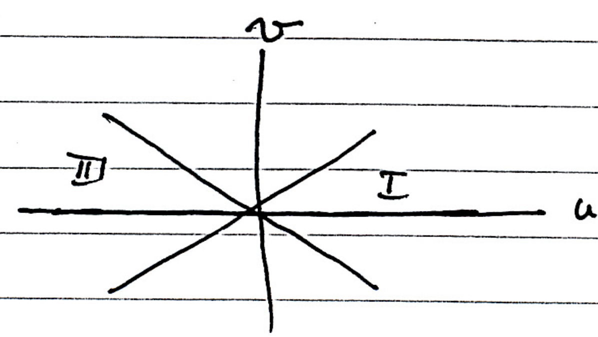
$$\lambda_p \simeq 1.62 \times 10^{-33}\ cm$$

what do we expect

- No true singularities
- quantum gravitational effects should prevent divergences of classical GR.
- while such a theory does not yet exist, semi-classical theories suggest that white holes are unstable and could not exist for more than a Planck time

$$t_p = \frac{\lambda_p}{c} = \left(\frac{\hbar G}{c^5}\right)^{1/2} \simeq 5.39 \times 10^{-44}\,s.$$

Worm holes



Consider spacelike hypersurface $v=0$, which extends from $u=-\infty$ to $u=+\infty$, in that case

$$ds^2 = -\frac{32(GM/c^2)^3}{r}\exp\left(-\frac{r}{\frac{2GM}{c^2}}\right)(-du^2) + r^2 d\Omega^2$$

Restrict geometry to plane $\theta = \pi/2$

$$\therefore\ ds^2 = \frac{32(GM/c^2)^3}{r}\exp\left(-\frac{r}{2GM/c^2}\right)du^2 + r^2 d\phi^2$$