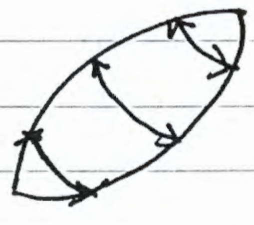
~~[illegible]~~ Curvature of Spacetime

Many local inertial reference frames, in which SR is locally valid, fitted together make up global structure of spacetime. Local manifestations of curvature add up to give appearance of long-range gravity

Gravitation does not show up in motion of one test particle, but only in change of separation between 2 or more test particles

- We can always transform away gravity in local inertial reference frame for 1 test particle

- But we cannot do so for relative separations of 2 test particles



- Curvature of spacetime causes freely falling bodies (rockets with motors off) to first separate and then come together along geodesic paths.

Source of Curvature: mass and momentum of massive bodies warp surrounding spacetime.

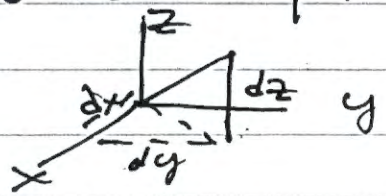
Metric: Key Ingredient in GR

Metric tells us how to compute distances separating objects in space or in spacetime. Contains all the information we need.

Spatial Metrics

3D flat Space Metric: ($dt=0$)

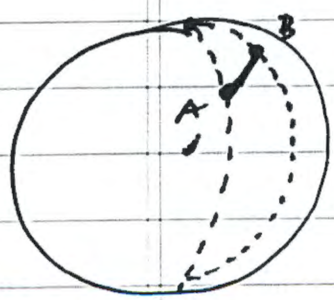
$$dS^2 = dx^2 + dy^2 + dz^2 \quad \text{(Pythagorean theorem)}$$



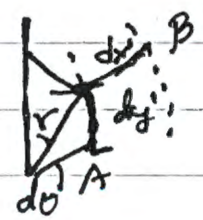
2D flat space Metric ($dt=0, dz=0$)

$$dS^2 = dx^2 + dy^2$$

2D Curved Space: Surface of a two-Sphere



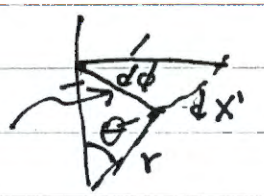
Differential displacement between A, B



$dy' = r\,d\theta$

$\cancel{dx'}$

$r\sin\theta$



$dx' = r\sin\theta\,d\phi$

$$dS^2 = dx'^2 + dy'^2 = r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

Pythagorean works because for small ~~ds~~ displacements

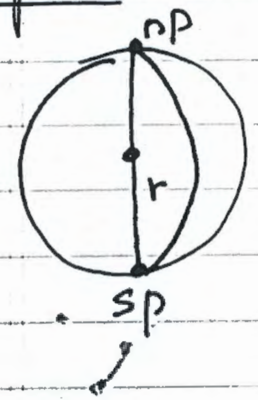
sphere is locally flat and Euclid works.
But for large changes (on order r), Euclid is incorrect

$$s_{Euc} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$ {gives wrong answer for displacements confined to sphere}

Correct answer

$$S = \int_{\theta_1,\phi_1}^{\theta_2,\phi_2} ds = r \iint_{\theta_1\phi_1}^{\theta_2\phi_2} \sqrt{d\theta^2+\sin^2\theta\, d\phi^2}$$

Example: Distance between north pole & south pole



Euclid: $S_{Euc} = 2r$

Spherical: Since $d\phi = 0$

$$S_{sphe} = r\int^{\pi} d\theta = \pi r$$

Ratio of circumference to radius for circle on sphere

Circle: Locus of points on a surface which are constant distance, "radius", along surface from fixed point

- Let np be the center of circle
- Circle is curve given by fixed polar angle θ

Circumference : Distance around curve $\theta = \Theta$.
In this case $d\theta = 0$

$$C = S = \int_0^{2\pi} r \sin\Theta \, d\phi = 2\pi r \sin\Theta$$

Radius : Distance from np along curve for which Θ varies, but $\phi = $ const., to circle

$$R = S = \int_0^{\Theta} r \, d\theta = r\,\Theta \Rightarrow \Theta = R/r$$

therefore $C = \cancel{2\pi r \sin\Theta} = 2\pi r \sin(R/r)$

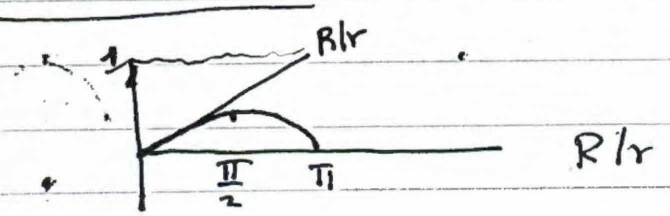
$$\boxed{\text{Ratio: } \frac{C}{R} = 2\pi \frac{\sin(R/r)}{(R/r)}}$$

• Limit : when radius of circle is small compared to radius of sphere: $R \ll r$

$$\frac{C}{R} \simeq 2\pi \times \frac{(R/r)}{(R/r)} = 2\pi$$

Get back Euclidean result

• In General : $\sin(R/r) < R/r$



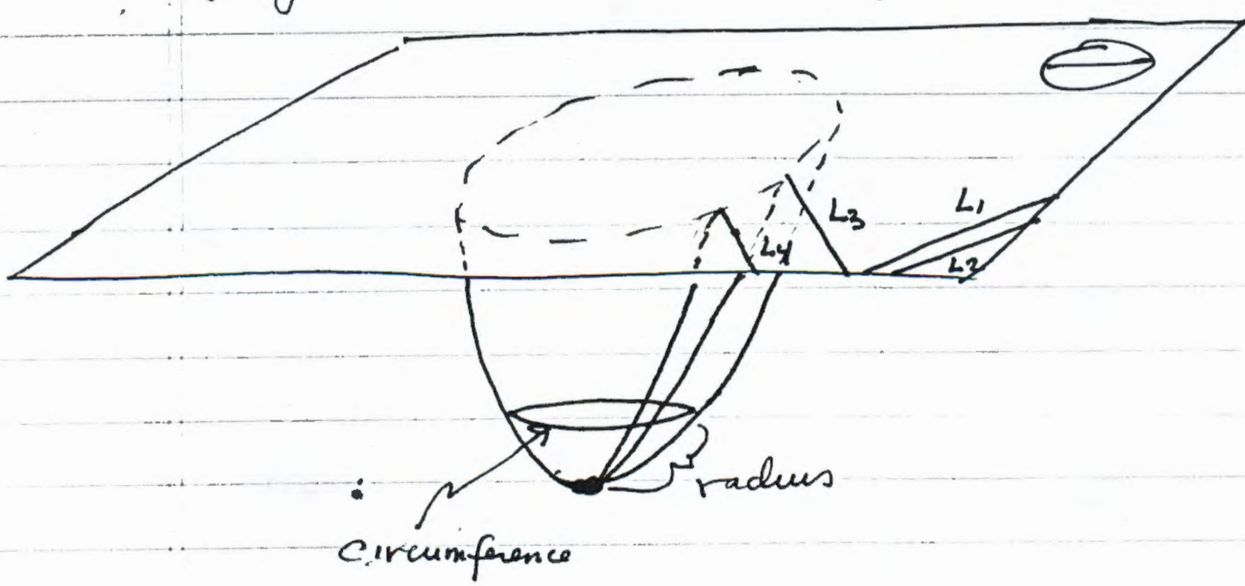
As a result: $C_{sphere} < C_{Euclid}$

What about 3D?

We can visualize 2D curved surfaces because we can embed them in a flat 3D Euclidean space. ~~[illegible]~~ How do we visualize 3D non-Euclidean space? Embed them in fictitious 4D Euclidean Space

Example

Consider a civilization of people inhabiting a 2D Universe: Their Universe is a curved bowl-like surface. They can "see" only with light signals propagating on surface. Both light & people move only along geodesic paths on surface



Geodesics: "Straight lines" are geodesics; i.e. shortest distances between points on space

- **Bottom:**

  Here surface is segment of a sphere. Thus "straight lines" are segments of great circles like equator or lines of constant longitude

- **Outside:**

  Outside lip of bowl, universe is flat

- **Parallel Lines:**

Outside: L_1, L_2 in flat space never cross: they know they are in flat space

Inside:
But L_3, L_4 start out parallel, but they converge in the bowl and cross at the bottom. They know space is curved.

- **Circles:**

Outside: measurements indicate $\frac{C}{r} = 2\pi$

Inside: $\frac{C}{r} < 2\pi$

- **Triangles**

outside 

$\alpha + \beta + \gamma = \pi$

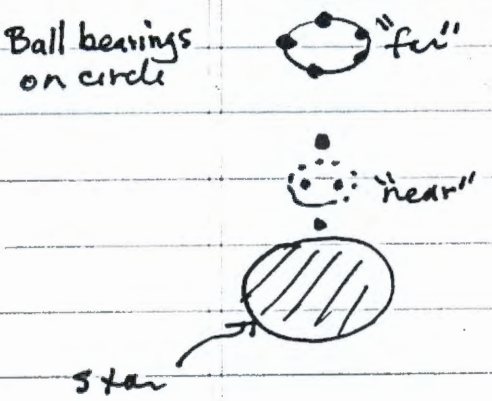
Inside 

$\alpha + \beta + \gamma > \pi$

Curvature and Mass + Momentum

GR takes mass and pressure within a system and predicts the metric. This is done with use of the Einstein field equations. In most cases mass dominates effects of pressure. In effect these equations tell us how curvature responds to mass.

(1) Measure of Curvature : Tidal Distortions



as ~~the~~ ball bearing test bodies approach star, evidence of curvature is in tidal distortion away from circle

~~can~~ transform away constant force, but not force gradient; i.e., tide

(2) Einstein Field Equations

$$\left(\text{Measure of Curvature}\right) = \left(\begin{matrix}\text{universal}\\\text{constant}\end{matrix}\right) * \left(\begin{matrix}\text{mass}\\\text{energy}\end{matrix}\right)$$

$$\underset{\substack{\uparrow\\\text{Ricci}\\\text{tensor}}}{R_{ab}} - \tfrac{1}{2} g_{ab} \cdot \underset{\substack{\uparrow\\\text{Ricci}\\\text{Scalar}}}{R} = \frac{8\pi G}{c^4} \times \underset{\substack{\uparrow\\\text{stress energy}\\\text{tensor}}}{T_{ab}}$$

~~Rab~~ $R = R_{ab} g^{ab}$: $R_{ab} = f(g_{ab})$

in weak field $(GM/Rc^2 \ll 1)$, slow motion $(v/c \ll 1)$ limit. these become

$$\nabla^2 \Phi = 4\pi G \rho \quad \text{(Poisson equation)}$$

Examples of famous metrics that are solutions to Einstein Field equations

(A) Friedmann/Robertson-Walker solution for the Universe

Universe filled with matter. Geometry is homogeneous and isotropic. In this case spacetime metric:

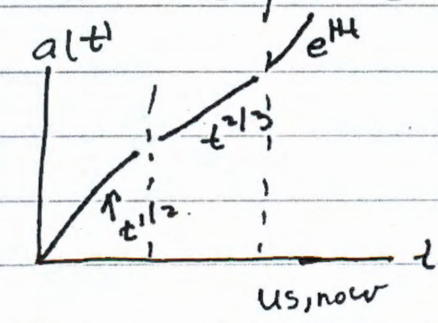
$$ds^2 = -c^2 dt^2 + dl^2$$

most general case: $dl^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$

where $k = -1, 0, +1$: for open, flat, closed Geometry. We now know $k=0$. Therefore I can write

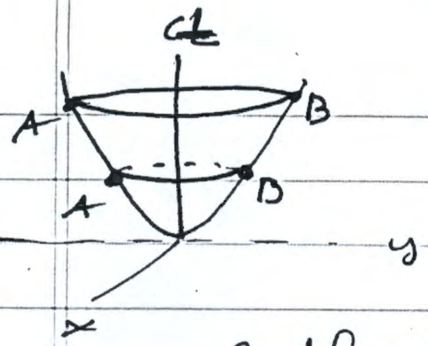
$$dl^2 = a^2(t)(dx^2 + dy^2 + dz^2)$$

$a(t)$ is the scale factor: Solution looks like:



Expansion History

Expanding Universe: Instantaneous space-like distances between galaxies increase with time.



Distance between Galaxies A & B:

$$l = a(t)\sqrt{(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2}$$

In this type of coordinate system x_A, x_B are labels. They stay fixed for galaxies A & B. What does change is $a(t)$, which we have seen, increases with time. A & B move away from each other. All objects recede from one another.

(B) Isolated Spherical Bodies

Metric is very different for an isolated localized object. In fact, since 1915 there are only 2 exact solutions to the Einstein field equations for localized objects; i.e., objects away from which space becomes asymptotically flat. One of these is for a spherically symmetric mass distribution: the Schwarzschild exterior solution: tells us how space is warped outside a spherical star, BH, with mass M.

Schwarzschild Metric

$$-ds^2 = \left(1 - \frac{2GM/c^2}{r}\right)(c\,dt)^2 - \frac{dr^2}{1-\frac{2GM/c^2}{r}} - r^2\overbrace{(d\theta^2+\sin^2\theta\, d\phi^2)}^{d\Omega^2}$$

(1) $r^2 = x^2 + y^2 + z^2$: usual radial coordinate

(2) $r^2 d\Omega^2$ is just constant r displacement on surface of 2D sphere, written in spherical coordinates

Furthermore

(3) Metric includes $c\,dt$. So it is full spacetime metric, not just spatial part dl^2

(4) Spherically Symmetric (Rotate coordinate system and metric remains invariant).

(5) Time-Independent : Unlike FRW metric, coefficients of differentials are independent of time. So this is a static solution that holds for all time.

(6) Curvature

Curved - Space:

$$dl^2_{\text{schwarz}} = \left(1 - \frac{2GM/c^2}{r}\right) dr^2 + r^2 d\Omega^2$$

$$dl^2_{\text{Euclid}} = dr^2 + r^2 d\Omega^2$$

Curved - Spacetime:

$$\text{SR: } ds^2 = -c^2 dt^2 + dl^2_{\text{Euclid}}$$

$$\text{Schwarz } ds^2 = -\left(1 - \frac{2GM/c^2}{r}\right) c^2 dt^2 + dl^2_{\text{schwa}}$$

We will study this metric exhaustively

Weirdness: Something strange happens when $r = \frac{2GM}{c^2}$

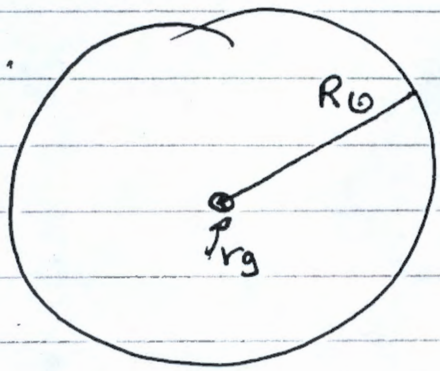
metric blows up there.

Recall: $r = r_g$ (Schwarzschild radius)

$$= \frac{2GM}{c^2} = 3\text{ km }(M/M_\odot)$$

Schwarzschild Radius

Sun.



$R_\odot = 7\times10^5$ km

$r_g = 3$ km for $M = M_\odot$

So: compute metric at surface of the sun at $r = r_\odot$. The terms

$$\frac{2GM/c^2}{r} = \frac{r_g}{r_\odot} \ll 1$$

Here metric is close to SR metric, since

$$ds^2 = -c^2dt^2 + dr^2 + r^2 d\Omega^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Black Hole

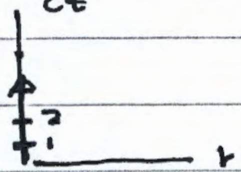
In this case mass is contained within r_g: in fact collapses within r_g, so effects near surface: say $r \simeq$ few $\times\, r_g$, GR is crucial!

Gravitational Time Dilation

Let's go back to Schwarzschild metric

$$-ds^2 = \left(1 - \frac{2GM/c^2}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2GM/c^2}{r}} - r^2 d\Omega^2$$

Consider the world line of a stationary clock; That is consider time intervals (ticks) between events for which $dr = 0$, $d\theta = 0$, $d\phi = 0$. Thus we are looking at timelike world line of a single stationary observer. In that case



$$-ds^2 = \left(1 - \frac{2GM/c^2}{r}\right)c^2dt^2$$

Recall definition of proper time interval: $c^2 d\tau^2 \equiv -ds^2$.

therefore: $c^2 d\tau^2 = -\left(1 - \frac{2GM/c^2}{r^2}\right)c^2 dt^2$

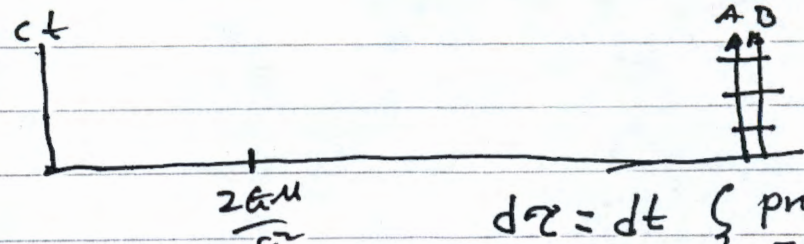
As a result: $d\tau = \sqrt{1 - \frac{2GM/c^2}{r}}\, dt$

or $dt = \dfrac{d\tau}{\sqrt{1 - \frac{2GM/c^2}{r}}}$

Gedanken Experiment

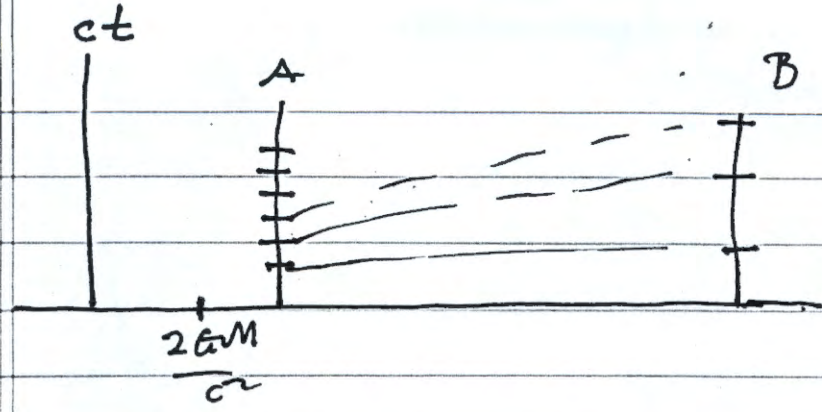
Take 2 clocks at rest at $r \gg 2GM/c^2$ and synchronize then

(1)

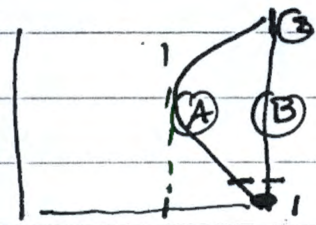


$d\tau = dt$ { proper time = coordinate time for both

(2) then Take clock A and slowly move it near $r \to 2GM/c^2$



Now ~~coordinate~~ proper time intervals are shorter compared to coordinate time intervals

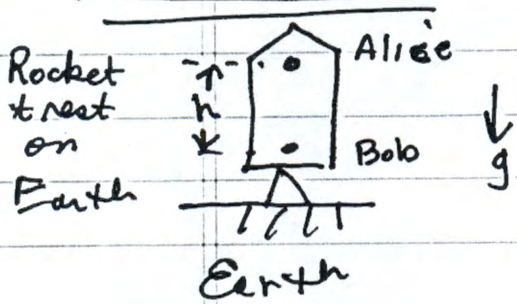


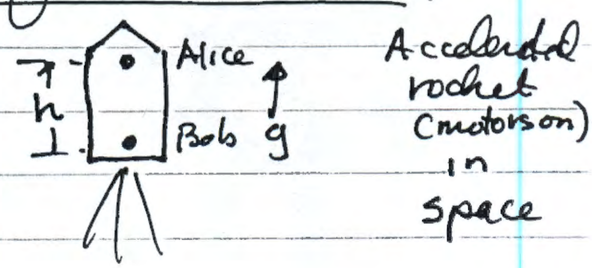
$$\tau_{12} = \int_{t_1}^{t_2} \sqrt{1 - \frac{2GM/c^2}{r(t)}}\, dt$$

clearly $\tau_{12} < t_2 - t_1$

Gravitational analogue of twin paradox. Shorter proper time has elapsed for (A) than for (B). Differs from time dilation in SR which is due only to speed of clock not its position is space.

Other Phenomena : Back to Equivalence Principle

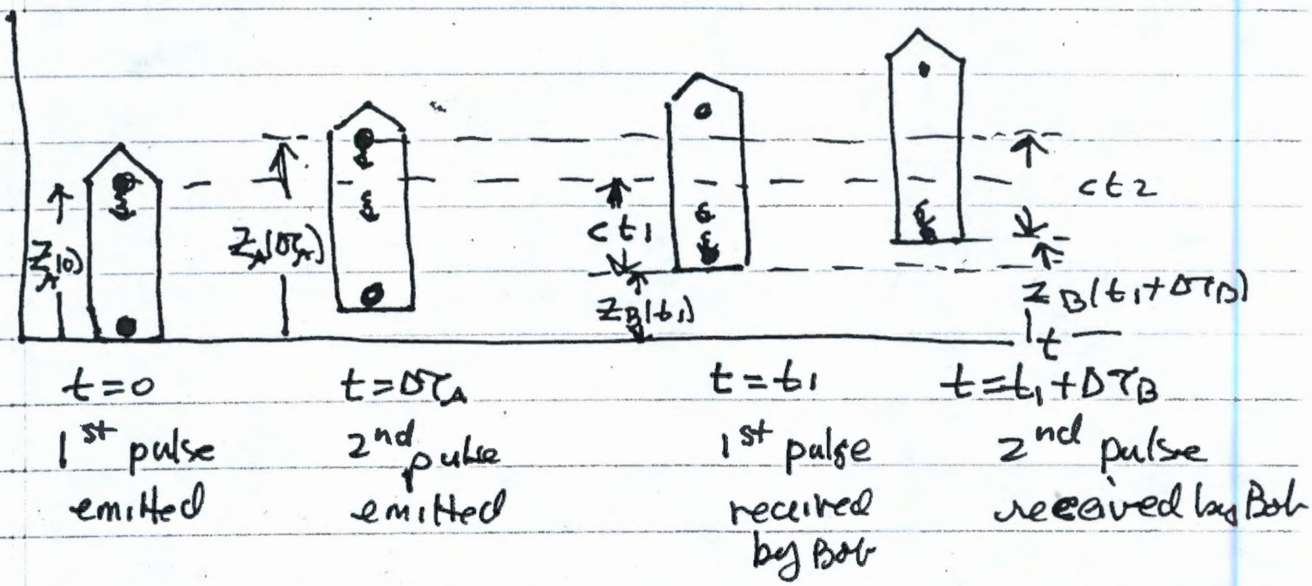




Accelerated Rocket

- Assume rocket accelerates uniformly along $+z$ axis
- at time t $\begin{cases} \bullet \text{ Bob is in tail } z_B(t) = \frac{1}{2} g t^2 \\ \bullet \text{ Alice is in nose } z_A(t) = \frac{1}{2} g t^2 + h \end{cases}$

- Alice has clock that emits light pulses separated by time intervals $\Delta\tau_A$. Bob receives them separated by $\Delta\tau_B$



- Distance traveled by first pulse

$$ct_1 = z_A(0) - z_B(t_1)$$

- Distance traveled by second pulse

$$ct_2 = c(\underbrace{t_1+\Delta\tau_B}_{\text{2nd pulse detected}} - \underbrace{\Delta\tau_A}_{\text{2nd pulse emitted}}) = z_A(\Delta\tau_A) - z_B(t_1+\Delta\tau_B)$$

Now insert appropriate expressions for z's

(1) $z_A(0) = h$; $z_B(t_1) = \frac{1}{2} g t_1^2$

therefore: $\boxed{ct_1 = h - \frac{1}{2} g t_1^2}$ (1)

(2) $z_A(\Delta\tau_A) = \frac{1}{2} g (\Delta\tau_A)^2 + h$

$$z_B(t_1 + \Delta\tau_B) = \frac{1}{2} g \,(t_1 + \Delta\tau_B)^2$$

$$c(t_1 + \Delta\tau_B - \Delta\tau_A) = \frac{1}{2} g (\Delta\tau_A)^2 + h - \frac{1}{2} g (t_1 + \Delta\tau_B)^2$$

<u>Assume</u>: $\Delta\tau_A, \Delta\tau_B \ll t_1$: Only terms linear in $\Delta\tau_A$, $\Delta\tau_B$ retained.

therefore: $c(t_1 + \Delta\tau_B - \Delta\tau_A) \simeq h - \frac{1}{2} g (t_1^2 + 2t_1\Delta\tau_B + \cdots)$

(3) or $\boxed{c(t_1 + \Delta\tau_B - \Delta\tau_A) = h - \frac{1}{2} g t_1^2 - g t_1 \Delta\tau_B}$

Subtract (1) from (3)

$$c(\Delta\tau_B - \Delta\tau_A) = -g t_1 \Delta\tau_B$$

Solve for $\Delta\tau_B$: $\Delta\tau_B (c + g t_1) = c\Delta\tau_A$

$$\therefore \Delta\tau_B = \frac{c\Delta\tau_A}{c + g t_1}$$

$$\Delta\tau_B = \frac{\Delta\tau_A}{1 + \frac{g t_1}{c}}$$

But from eq. ① we have:

$$h = ct_1 + \frac{1}{2} g t_1^2 = ct_1 \left[1 + \frac{g t_1}{2c} \right]$$

But for reasonable time intervals $g t_1 \ll c$
So to good approximation we have
$$t_1 \simeq h/c$$

Therefore: $$\Delta \tau_B \simeq \frac{\Delta \tau_A}{1 + \frac{gh}{c^2}}$$

or $$\boxed{\Delta \tau_B \simeq \Delta \tau_A \left(1 - \frac{gh}{c^2}\right)} \quad \left(\text{since } gh \ll c^2\right)$$

* Time interval in which pulse received is less than time interval in which pulse is emitted.

Equivalence Principle: Same effect occurs for stationary rocket in uniform gravitational field.

Frequency Shifts

Rate of emission (i.e., #pulses per second) $\nu_A = 1/\Delta \tau_A$
Rate of reception $\nu_B = 1/\Delta \tau_B$

$$\frac{1}{\Delta \tau_B} = \frac{1}{\Delta \tau_A \left(1 - \frac{gh}{c^2}\right)}$$

Or $\nu_B = \nu_A \left(1 + \frac{gh}{c^2}\right)$ ∴ frequency increases

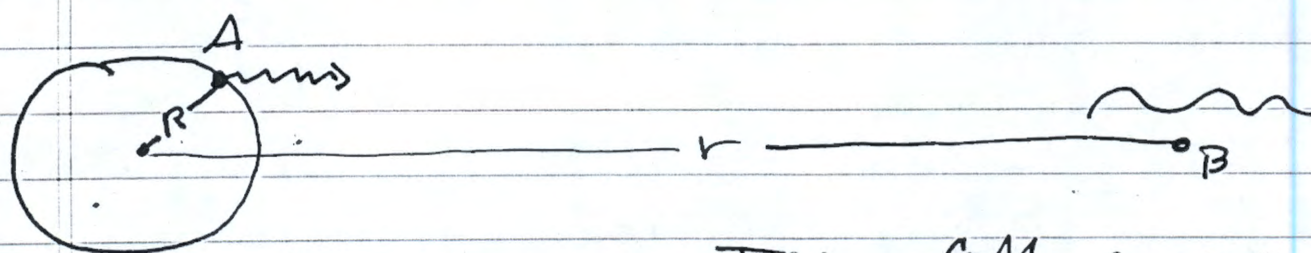
But $gh = \Phi_A - \Phi_B$

where Φ_A and Φ_B are gravitational potentials at Alice and Bob.

$$\boxed{\nu_B = \nu_A\left[1 + \frac{\overset{\text{emitter}}{\Phi_A} - \overset{\text{receiver}}{\Phi_B}}{c^2}\right]}$$

ν increase ⇒ λ decrease ⇒ blueshift

Redshift: Suppose light emitted from surface of a star with radius R and is detected at $r \gg R$



For spherical mass: $\Phi(r) = -\frac{GM}{r}$ for $r \geq R$

For $r = R$: $\Phi_A(R) = -\frac{GM}{R}$

For $r \gg R$: $\Phi_B \simeq 0$

$$\therefore \cancel{\boxed{\nu_B \Delta\nu}}\ \nu_B(r) = \nu_A(R)\left[1 + \frac{\Phi_A(R) - \Phi_B(r)}{c^2}\right]$$

$$\nu_B(r) = \nu_A(R)\left[1 + \frac{-\frac{GM}{R} - 0}{c^2}\right]$$

$$\boxed{\nu_B(r) = \nu_A(R)\left[1 - \frac{GM}{Rc^2}\right]}$$

In this case $\nu_B(r) < \nu_A(R)$ ∴ ν decreases and photon is redshifted to longer wavelength.

- Photon energy $E = h\nu$. Think of photons doing work as they "climb" out of strong field to weaker field. They expend energy $E = h\nu$; so ν decreases

Note: This is opposite of Rocket ship where photons traveled from weaker field (Alice) to stronger field (Bob). In the rocket-ship case, grav. field does positive work on photons & thereby increasing E and ν.

White Dwarfs: Redshifts are detected

$M_{WD} \simeq M_{\odot}$, $R_{WD} = 10^8$ cm; $R_{\odot} = 7\times10^{10}$ cm

$$\frac{\nu_B(r) - \nu_A(R)}{\nu_A(R)} = -\frac{GM}{Rc^2} = -\frac{6.7\times10^{-8}\cdot 2\times10^{33}}{10^8\times(3\times10^{10})^2} \simeq 10^{-3}$$

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8H-13
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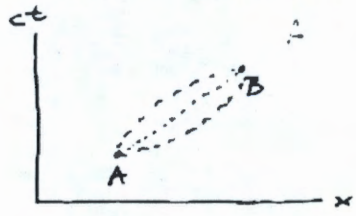
Geodesics and Extremal curves

Suppose we have 2 events in spacetime:



There are many possible paths connecting A and B. Along each path let's compute the interval

$$S = \int_A^B ds$$

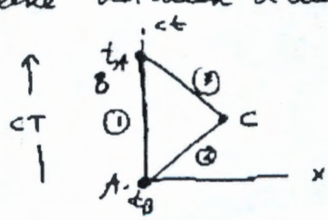
where value of S changes from one path to the next.

Geodesic: Geodesics are the gravitational free-fall orbits i.e., free-falls are those world lines that produce extremal values of S: (extremals are minimum, maximum, or saddle points)

Stated Differently: Free fall geodesic is curve that gives smallest (or largest) value of S

FLAT SPACE EXAMPLE

In flat space, interval measured along a straight timelike worldline between 2 events is maximum path between events.



$$dS^2 = dx^2 - c^2 dt^2$$

Define $c^2 d\tau^2 = -ds^2 = c^2 dt^2 - dx^2$

$$c\,\tau_{AB} = \int_A^B ds = \int \sqrt{c^2 dt^2 - dx^2}$$

- For path 1, $dx = 0$

Therefore: $c\Delta\tau_1 = \int_0^{cT} d(ct)$ or

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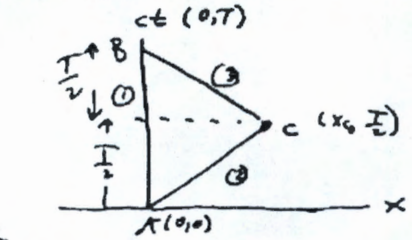
BH-6-7

Therefore $c\Delta\tau_1 = c(t_B - t_A) = cT$

- Now consider alternate route $A \to C \to B$. Such that

$\frac{dx}{dt} = +v; (A \to C)$

$\frac{dx}{dt} = -v; (C \to B)$



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$$c\Delta\tau_2 = \int_0^{T/2} \sqrt{c^2 dt^2 - dx^2} = \int_0^{T/2} \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2}\, dt =$$

$$c\Delta\tau_2 = \int_0^{T/2} \sqrt{c^2 - v^2}\, dt \quad ; \Rightarrow c\Delta\tau_2 = \sqrt{c^2 - v^2} \cdot \frac{T}{2}$$

- 3 Similarly: $c\Delta\tau_3 = \int_{T/2}^{T} \sqrt{c^2 - v^2}\, dt = \sqrt{c^2 - v^2}\,\frac{T}{2}$

$$\therefore |\Delta s_{2+3}| = \sqrt{c^2 - v^2} \cdot T = \sqrt{1 - \left(\frac{v}{c}\right)^2}\, cT$$

Clearly $|\Delta s_{2+3}| < |\Delta s_1|$ ∴ st. world line is extremal

Back to General Case: The idea is to take extremal values of the definite integral s (in curved spacetime)

~~3D $\delta s = 0$ → Calculus variations~~

~~→ Euclidean (Really dl) $s = \int_{x_A}^{x_B} \sqrt{dx^2 + dy^2 + dz^2}$ (space-like interval)~~

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So we desire curves $x(t)$ that produce ~~extreme~~ extrema in the interval s

Let $S = \int_{t_A}^{t_B} \left(\frac{ds}{dt}\right) dt$

Let $\frac{ds}{dt} = f$, where f is a function

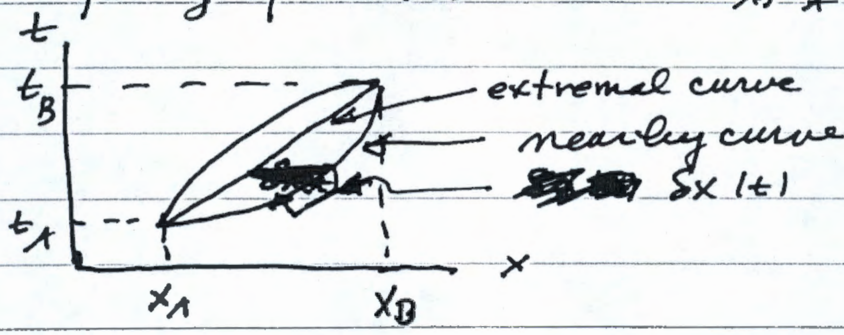
Flat Space 1D, $ds^2 = dx^2 \Rightarrow \frac{ds}{dt} = f(x')$ where $x' = \frac{dx}{dt}$

But, in spherical coordinates, and in more general case

$$f = f(x, x'; t) \quad \text{or 3D} \quad f(x, y, z; x', y', z'; t)$$

Back to 1-D

Consider a family of curves between $x_A, t_A \rightarrow x_B, t_B$



Then small variation induced by going from extremal curve to nearby curve by change δx should result in $\delta S = 0$. Thus if

$$S[x(t)] = \int_{t_A}^{t_B} f(x, x'; t)\, dt,$$

Extrema of S are defined by vanishing of its 1^{st} ~~derivative~~ order variation $\delta S[x(t)]$ for arbitrary $\delta x(t)$ along path $[x_A, t_A] \rightarrow [x_B, t_B]$

Let $\delta S = S[x+\delta x] - S[x]$

Therefore $S[x+\delta x] = \int_{t_A}^{t_B} f(x+\delta x, \dot{x}+\delta\dot{x}, t)\,dt$ ∴

Make Taylor Series expansion around $x, \dot{x}$, retaining linear terms only

$$f(x+\delta x, \dot{x}+\delta\dot{x}, t) = f(x,\dot{x},t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta\dot{x}$$

Therefore $$\delta S = \int_{t_A}^{t_B} \left\{ f(x,\dot{x},t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta\dot{x} - f(x,\dot{x},t) \right\} dt$$

$$\delta S = \int_{t_A}^{t_B} \left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta\dot{x} \right] dt$$

<u>Integrate 2nd term by parts:</u>

$$\int_{t_A}^{t_B} \Big[\underset{u}{\frac{\partial f}{\partial \dot{x}}} \underset{dv}{\delta\dot{x}} \Big] dt = \left[\frac{\partial f}{\partial \dot{x}}\delta x \right]_{t_A}^{t_B} - \int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\delta x\,dt$$

(note $\delta\dot{x} = \frac{d}{dt}\delta x) = \delta\left(\frac{dx}{dt}\right)$)

Since $\delta x(t_B) = \delta x(t_A) = 0$ on boundaries, $[\ ] = 0$

As a result: $$\delta S = \int_{t_A}^{t_B} \left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) \right] \delta x\,dt = 0$$

Thus for arbitrary δx, integrand must vanish

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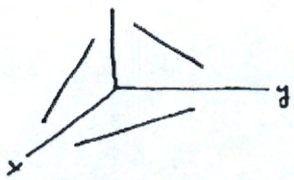
$$\frac{dz}{dx} = m_2 \Rightarrow z = m_2 x + b_2$$



straight lines, as expected.

$$\frac{dz}{dy} = m_3 \Rightarrow z = m_3 y + b_3$$

$\boxed{\text{MAX}}$ Straight lines are geodesics in Euclidean space.

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$[\quad] = 0$ or

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0}$$ Lagrange eq.

f satisfying above equation is a necessary condition for S to be extremal been A + B.

Examples (spatial metric)

$$S = \int_a^b \sqrt{dx^2+dy^2+dz^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt$$

In this case t is not time but parameter varying along curves

$f = (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}$, so f independent of x, y, z.

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial f}{\partial x} = 0 \;;\quad \frac{\partial f}{\partial \dot{x}} = \frac{1}{2}\frac{2\dot{x}}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}}$$

Therefore Lagrange equations ⇒

$$0 - \frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0 \Rightarrow \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_1 \quad (1)$$

Same must be true for y, z $$\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_2 \quad (2)$$

$$\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = c_3 \quad (3)$$

Divide (2)/(1)

$$\frac{dy/dt}{dx/dt} = \frac{c_2}{c_1} \Rightarrow \frac{dy}{dx} = m_1$$

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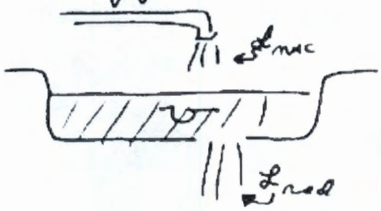
Week #4 Lecture 8 Endpoints of Stellar Evolution [BH-54]

Because luminosity of the sun, $\mathcal{L} = 4\times10^{33}$ ergs s^{-1},

$$\Delta t = \frac{3.6\times10^{48}}{4\times10^{33}} \simeq .9\times10^{15}\,s \simeq \frac{.9\times10^{15}}{3\times10^{7}} \simeq 3.0\times10^{7}\text{ years}$$

But presence of ancient trilobytes shows that sun has been at roughly present luminosity for a much longer time, probably throughout age of the earth, 4.5×10^{9} y.

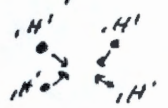
Solution: We need an energy faucet to constantly supply energy at rate by which it is radiated.



net energy change: $\frac{dE}{dt} = \mathcal{L}_{nuc} - \mathcal{L}_{rad} \simeq 0$

Nuclear Fusion:

Bethe and others showed that core of sun was hot enough ($T\sim10^{7}$K) and sufficiently dense ($\rho \approx 50$ g cm^{-3}) for fusion reactions to supply required energy: [Barrier penetration:]



$\longrightarrow {}_2He^4 + Q$

$$4({}_1H^1) \longrightarrow {}_2He^4 + Q$$

$$Q \simeq 4(m_p c^2) - (m_{He} c^2) \simeq 26\text{ MeV} \;;\; \epsilon = \frac{Q}{4m_H} = 6\times10^{18}\,\frac{\text{erg}}{\text{g}}$$

conversion of mass $\to$ energy

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