Physics 4E — **Final Exam Solutions** — Spring 2012

Problem 1

(a) The K_α photons result from transitions from $n=2$ to $n=1$, so

$$\frac{hc}{\lambda} = E_0(Z-1)^2\left(1-\frac{1}{4}\right) \quad , \; E_0 = 13.6\,eV \quad \Rightarrow$$

$$(Z-1)^2 = \frac{hc}{\lambda E_0}\cdot\frac{4}{3} \Rightarrow Z = 1 + \left(\frac{12{,}400}{0.4328 \times 13.6}\times\frac{4}{3}\right)^{1/2} \Rightarrow \boxed{Z=54}$$

(b) For Compton scattering,

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta)$$

largest wavelength this $\theta = \pi \Rightarrow \lambda' = \lambda + \frac{2h}{m_e c} \Rightarrow \boxed{\lambda' = 0.2485\,\mathring{A}}$

smallest wavelength is $\theta = 0 \Rightarrow \boxed{\lambda' = 0.2\,\mathring{A}}$

(c) $$E_{kin}^{electron} = -\frac{hc}{\lambda'(\theta=\pi)} + \frac{hc}{\lambda} = 12{,}107\,eV$$

Taking into account the work function, the kin energy is $\boxed{12{,}103\,eV}$

(d) The highest possible energy is in a process like in the photoelectric effect, so $E_{kin} = \frac{hc}{\lambda} - \phi = 62{,}000\,eV - 4\,eV = \boxed{61{,}996\,eV}$

(e) If the incoming photon is absorbed by an electron in the $n=1$ shell and that electron is ejected, a K_α emission results as discussed in (a). The ejected electron has kinetic energy

$$E_{kin} = \frac{hc}{\lambda} - E_0\frac{Z^2}{n^2} = 62{,}000\,eV - 39{,}657.6\,eV = 22{,}342\,eV$$

or more accurately, taking into account ϕ, $\boxed{E_{kin} = 22{,}338.4\,eV}$

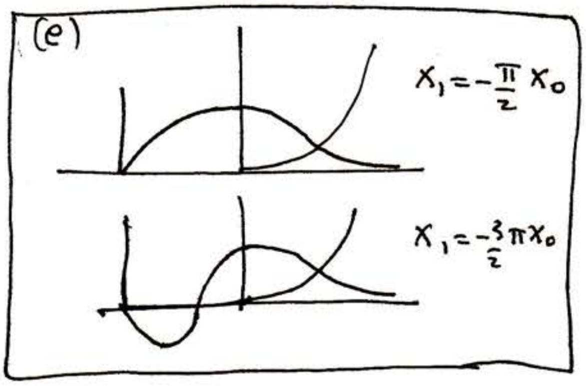
Problem 2

(a) $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$; $\psi = Ae^{-\lambda x^2}$

$\frac{\partial \psi}{\partial x} = -2\lambda x e^{-\lambda x^2}$; $\frac{\partial^2 \psi}{\partial x^2} = -2\lambda e^{-\lambda x^2} + 4\lambda^2 x^2 e^{-\lambda x^2} \Rightarrow$

$\frac{\hbar^2}{m}\lambda - \frac{2\hbar^2}{m}\lambda^2 x^2 + \frac{\hbar^2}{2m x_0^4}x^2 = E \Rightarrow$

$\frac{2\cancel{\hbar^2}}{\cancel{m}}\lambda^2 = \frac{\cancel{\hbar^2}}{2\cancel{m}x_0^4} \Rightarrow \lambda^2 = \frac{1}{4x_0^4} \Rightarrow \boxed{\lambda = \frac{1}{2x_0^2}}$



(b) From eq. above

$E = \frac{\hbar^2}{m}\lambda \Rightarrow \boxed{E = \frac{\hbar^2}{2m x_0^2}}$

(c) For $-x_1 < x < 0$, $V(x) = 0$, so

$\psi(x) = B\sin kx + C\cos kx$

At $x = 0$, $\psi'(x) = 0$ since $\frac{\partial \psi}{\partial x} = -2\lambda x e^{-\lambda x^2}$ for $x \geq 0$.

Therefore, $B = 0$. From continuity at $x = 0$, $\boxed{C = A}$

The energy is $E = \frac{\cancel{\hbar^2}k^2}{2\cancel{m}} = \frac{\cancel{\hbar^2}}{2\cancel{m}x_0^2} \Rightarrow \boxed{k = \frac{1}{x_0}} \Rightarrow$

$$\boxed{\psi(x) = A\cos\left(\frac{x}{x_0}\right)}$$

(d) At x_1, $V = \infty \Rightarrow \psi(x_1) = 0 \Rightarrow \frac{x_1}{x_0} = -\frac{\pi}{2}, -\frac{3}{2}\pi, \dots$

$\Rightarrow \boxed{x_1 = -\frac{\pi}{2}x_0}$, $\boxed{x_1 = -\frac{3}{2}\pi x_0}$ are 2 possible values.

(e) See graph above

Problem 3 $\qquad \psi(x) = Bxe^{-x/2x_0}, \quad x_0 = 2\,\text{Å}$

(a) $$1 = \int_0^\infty dx\, B^2 x^2 e^{-x/x_0} = B^2 \frac{2!}{\frac{1}{x_0^3}} = 2x_0^3 B^2 \Rightarrow B = \frac{1}{(2x_0^3)^{1/2}}$$

(b) $$\langle x \rangle = \frac{\int_0^\infty dx\, x^3 e^{-x/x_0}}{\int_0^\infty dx\, x^2 e^{-x/x_0}} = \frac{3!\, x_0^4}{2!\, x_0^3} = \boxed{3x_0 = 6\,\text{Å}}$$

(c) $$\langle x^2 \rangle = \frac{\int_0^\infty dx\, x^4 e^{-x/x_0}}{\int_0^\infty dx\, x^2 e^{-x/x_0}} = \frac{4!\, x_0^5}{2!\, x_0^3} = 12x_0^2 \Rightarrow$$

$$\Delta x = \sqrt{12-9}\; x_0 = \boxed{\sqrt{3}\, x_0 = 1.732 x_0 = 3.46\,\text{Å}}$$

(d) $$p\psi = \frac{\hbar}{i}\frac{d}{dx} Bxe^{-x/2x_0} = \frac{\hbar}{i} B\left(e^{-x/2x_0} - \frac{x}{2x_0} e^{-x/2x_0}\right)$$

$$\langle p \rangle = \int dx\, \psi^* p \psi = \frac{\hbar}{i} B^2 \int_0^\infty dx \left(xe^{-x/x_0} - \frac{x^2}{2x_0} e^{-x/x_0}\right) =$$

$$= \frac{\hbar}{i} B^2 \left(1!\, x_0^2 - \frac{2!}{2}\frac{x_0^3}{x_0}\right) \boxed{= 0}$$

(e) $$\langle p^2 \rangle = \left\langle -\hbar^2 \frac{d^2}{dx^2} \right\rangle \; ; \; \frac{d^2}{dx^2} xe^{-x/2x_0} = -\frac{2}{2x_0} e^{-x/2x_0} + \frac{x}{4x_0^2} e^{-x/2x_0}$$

$$\langle p^2 \rangle = \hbar^2 \frac{\int_0^\infty dx \left(\frac{x}{x_0} - \frac{x^2}{4x_0^2}\right) e^{-x/x_0}}{\int_0^\infty dx\, x^2 e^{-x/x_0}} = \frac{\hbar^2 \left(1!\, x_0 - \frac{2!}{4} x_0\right)}{2!\, x_0^3} = \frac{\hbar^2}{4x_0^2}$$

$$\Rightarrow \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{2x_0} \; ; \; \boxed{\Delta p \Delta x = \frac{\hbar}{2x_0} \cdot \sqrt{3} x_0 = \frac{\sqrt{3}}{2}\hbar \sim \hbar \text{ o.k.}}$$

(f) $$\langle E_{kin} \rangle = \langle p^2 \rangle / 2m_e = \boxed{\hbar^2 / 8 m_e x_0^2 = 0.238\,\text{eV}}$$

Problem 4

(a) $\psi(x) = A \sin kx + B \cos kx$

Since $\psi(x=0) = 0 \rightarrow B = 0 \Rightarrow$

$\boxed{\psi(x) = A \sin kx}$, $E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$

Since $E = 2eV \rightarrow \boxed{k = 0.7245 \text{ Å}^{-1}}$

(b) $\boxed{\psi(x) = C e^{-\alpha x}}$, $\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

$\Rightarrow \alpha = \sqrt{\frac{2}{7.62}(6-2)} \text{ Å}^{-1} \Rightarrow \boxed{\alpha = 1.025 \text{ Å}^{-1}}$

(c) By continuity of ψ and ψ',

$A \sin(kL) = C e^{-\alpha L}$
$kA \cos(kL) = -\alpha C e^{-\alpha L}$ $\Rightarrow \frac{1}{k}\tan(kL) = -\frac{1}{\alpha} \Rightarrow$

$\rightarrow \tan(kL) = -\frac{k}{\alpha} = -0.7068 \rightarrow kL = 2.526 \Rightarrow \boxed{L = 3.487 \text{ Å}}$

(d) $\boxed{E_1 = \frac{\hbar^2 \pi^2}{2mL^2} = 3.09 \text{ eV}}$

(e) $2eV = \frac{\hbar^2 \pi^2}{2m L_{eff}^2} \rightarrow \boxed{L_{eff} = 4.33 \text{ Å}}$

(f) The "penetration" of the wavefunction in the "forbidden region" where $V_0 > E$ is $\sim \alpha^{-1} = \frac{1}{1.025} \text{ Å} = 0.976 \text{ Å}$.

If we add that to $L = 3.487$ Å, we get 4.46 Å. That is pretty close to the effective length $L_{eff} = 4.33$ Å.

Problem 5

(a) ionization energy is $\frac{E_0 Z^2}{n^2} \equiv I$. $\lambda = 80\,\text{Å}$, $\frac{hc}{\lambda} = 155\,eV$

$$E_{kin} = \frac{hc}{\lambda} - I \Rightarrow 32.6\,eV = 155\,eV - E_0 \frac{Z^2}{n^2} \Rightarrow$$

$$\frac{Z^2}{n^2} = \frac{155 - 32.6}{13.6} = 9 \Rightarrow \boxed{\frac{Z}{n} = 3}$$

$$\Rightarrow \boxed{n=1,\ Z=3\ ;\ \text{or}\ n=2,\ Z=6\ ;\ \text{or}\ n=3,\ Z=9\ ;\ \ldots}$$

(b) Since a magnetic field gives 3-fold splitting $\Rightarrow l = 1$.

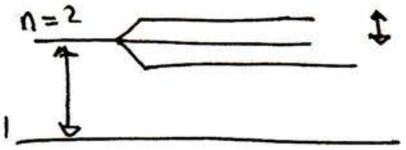
The smallest n is $n = 2 \Rightarrow \boxed{Z = 6}$

$\boxed{\text{Quantum numbers are } n = 2,\ l = 1,\ m_l = 1, 0, -1.}$

Wavefunctions:

$$\boxed{\begin{aligned} \psi(r,\theta,\phi) &= C e^{-3r/a_0}\, r \sin\theta\, e^{i m_l \phi} \quad , m_l = \pm 1 \\ &= C e^{-3r/a_0}\left(1 - \frac{3r}{a_0}\right)\cos\theta \quad , m_l = 0 \end{aligned}}$$

(c)



$$\Delta E = E_0 Z^2 \left(\frac{3}{4}\right) \pm \mu_B B = 367.2\,eV \pm 0.116\,eV$$
$$\qquad\qquad\qquad\qquad\ \ 0 \qquad\qquad\qquad\qquad\ \ 0$$

$$\Delta E = \frac{hc}{\lambda} \Rightarrow \boxed{\begin{aligned} \lambda_1 &= 33.76\,\text{Å} \\ \lambda_2 &= 33.77\,\text{Å} \\ \lambda_3 &= 33.78\,\text{Å} \end{aligned}}$$

(d) Spin-orbit splitting $\propto Z^4$, according to quiz 4, prob 3.
For $Z = 1$, it is $4.5 \times 10^{-5}\,eV$. $Z^4 = 6^4 = 1296$

$\Rightarrow$ spin-orbit splitting here is $\sim \boxed{5.8 \times 10^{-2}\,eV}$

Problem 6

$$kT_E = \hbar\omega \Rightarrow T_E = \frac{0.15\,eV}{\frac{1}{11{,}600}\,eV}\ °K \Rightarrow \boxed{T_E = 1740°K}$$

(b) $$\langle E\rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

For $T \gg T_E$, $\langle E\rangle = kT \Rightarrow \boxed{\langle E\rangle = 86.2\,eV}$

(c) at $T = 0$, $\boxed{E = \frac{\hbar\omega}{2} = 0.075\,eV}$

(d) $$\langle E\rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{1} - 1} = \hbar\omega(0.5 + 0.582) = 1.082\,\hbar\omega$$

$$\Rightarrow \boxed{\langle E\rangle = 0.162\,eV}$$

(e) Boltzmann factor is $e^{-E/kT}$

$\Rightarrow$ ratio of probabilities is:

$$\frac{P(E_2)}{P(E_1)} = e^{-(E_2 - E_1)/kT} = e^{-0.3\,eV/0.15\,eV} = e^{-2} = 0.1353$$

$\Rightarrow$ if there are 10,000 oscillators with energy $E_1 = 1.575\,eV$,

$\boxed{\text{there are 1353 oscillators with energy } E_2 = 1.875\,eV}$

Problem 7

(a) $E_1 = \dfrac{\hbar^2 \pi^2}{2mL^2} \Rightarrow \dfrac{1}{m_p L_1^2} = \dfrac{1}{m_e L_2^2} \Rightarrow$

$$\Rightarrow \frac{L_2}{L_1} = \left(\frac{m_p}{m_e}\right)^{1/2} = \left(\frac{938.3}{0.511}\right)^{1/2} \Rightarrow \boxed{\frac{L_2}{L_1} = 42.9}$$

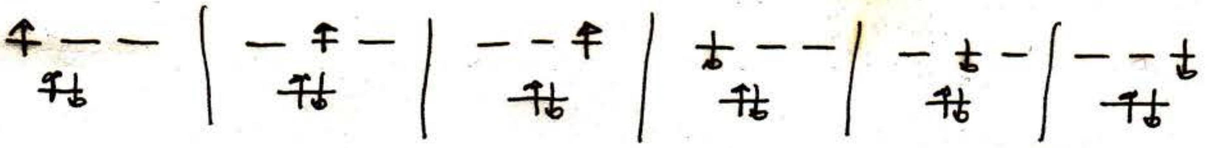
(b) $$T = e^{-2\sqrt{\frac{2m_e}{\hbar^2}(V_0 - E)}} = e^{-2\sqrt{\frac{9}{3.81}}} = e^{-3.07} = 0.0462$$

$\Rightarrow$ $\boxed{\text{46 electrons tunnel for every 1000 incident}}$

(c) Lowest state: E_{111}, next: $E_{211} = E_{121} = E_{112}$

Each state can accommodate 2 electrons of opposite spin.

Different states:



$\boxed{\text{there are 6 different states.}}$