Principle of Maximum Likelyhood, Propagation of Uncertainties for Racket Balls and Rods

Lecture # 5

Physics 2BL

Spring 2012

Outline

- Review of Gaussian distributions and rejection of data?
- Uncertainties for lab 2
  - Propagate errors
  - Minimize errors

Schedule

Meeting	Experiment
1 (Apr 2-6)	None (start Taylor)
2 (Apr 9-13)	1
3 (Apr 16-20)	1
4 (Apr 23-27)	2
5 (Apr 30-May4)	2
6 (May 7-11)	3
7 (May 14-18)	3
8 (May 21-25)	4
9 (May 28-June 1)	4
10 (June 4-8)	FINAL

Clicker Question 6

What is the correct way to report 653 ± 55.4 m

(a) 653.0 ± 55.4 m
(b) 653 ± 55 m
(c) 650 ± 55 m
(d) 650 ± 60 m

Keep one significant figure

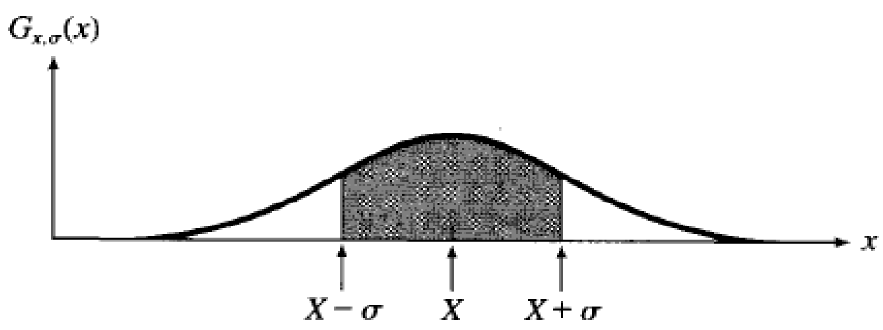
Last sig fig of answer should be same order of magnitude as error

The Gauss, or Normal Distribution

normalize $e^{-(x-X)^2/2\sigma^2}$ $\longrightarrow$ $\int_{-\infty}^{+\infty} f(x)dx = 1$

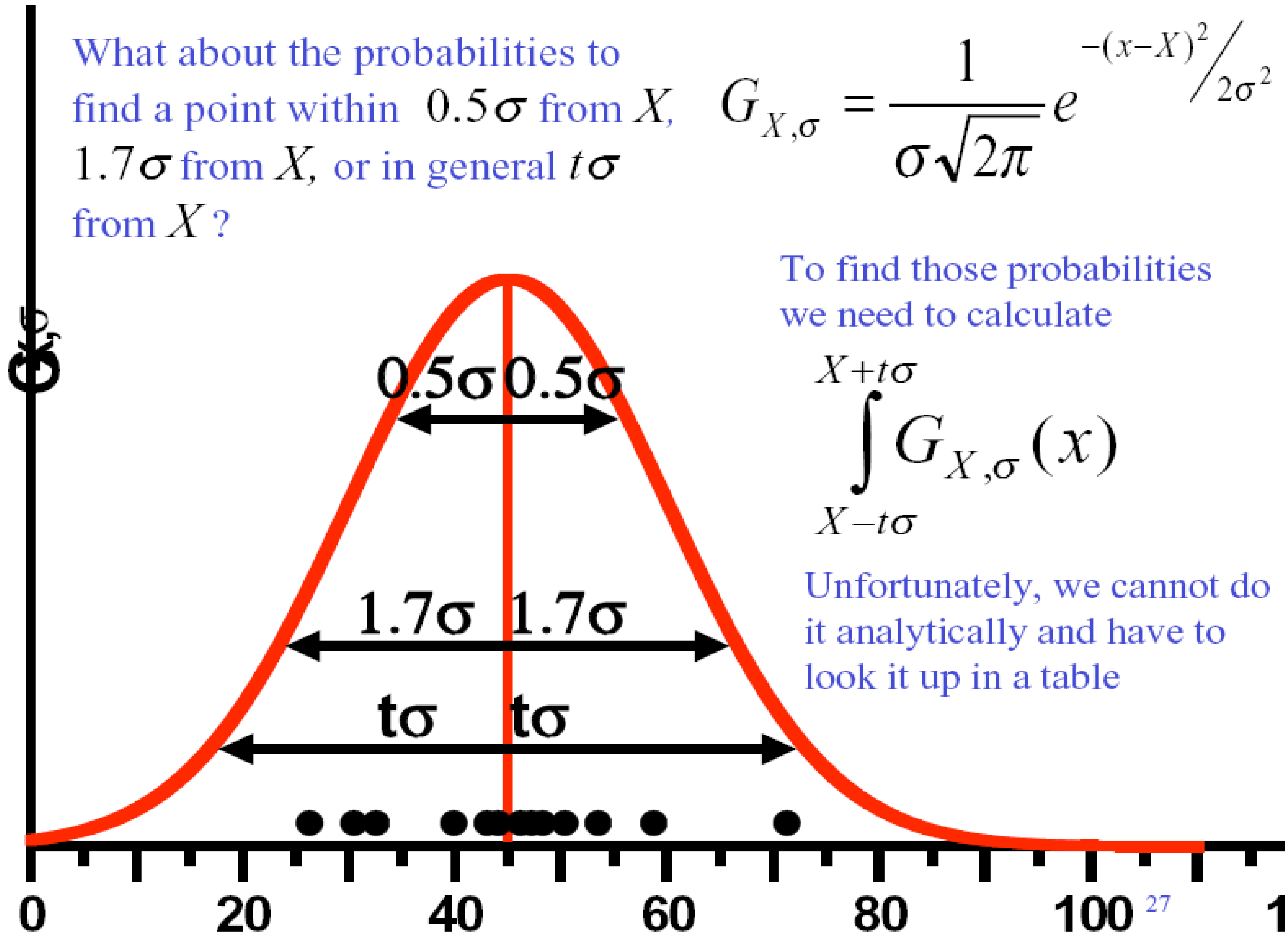
$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

standard deviation σ_x = width parameter of the Gauss function σ
the mean value of x = true value X



What about the probabilities to find a point within 0.5σ from X, 1.7σ from X, or in general $t\sigma$ from X?

$$G_{X,\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2 / 2\sigma^2}$$

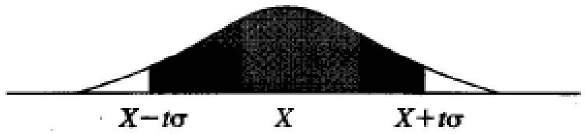


To find those probabilities we need to calculate

$$\int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)$$

Unfortunately, we cannot do it analytically and have to look it up in a table

Table A. The percentage probability, $Prob(\text{within } t\sigma) = \int_{X-t\sigma}^{X+t\sigma} G_{X,\sigma}(x)\,dx$, as a function of t.



t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80

$t = 1$

p. 287 Taylor

Change to rubric for Exp 2

Calculate the expected number of trials that should exceed your average time by one standard deviation and compare it to what you observe.

Clicker Question 7

What is your age?

(a) ≤ 18
(b) 19,20
(c) 21,22
(d) 23,24
(e) ≥ 25

Compatibility of a measured result(s): t-score

- Best estimate of x: $x_{best} \pm \sigma_{\overline{X}}$

- Compare with expected answer x_{exp} and compute t-score:

$$t \equiv \frac{\left| x_{best} - x_{\exp ected} \right|}{\sigma_X}$$

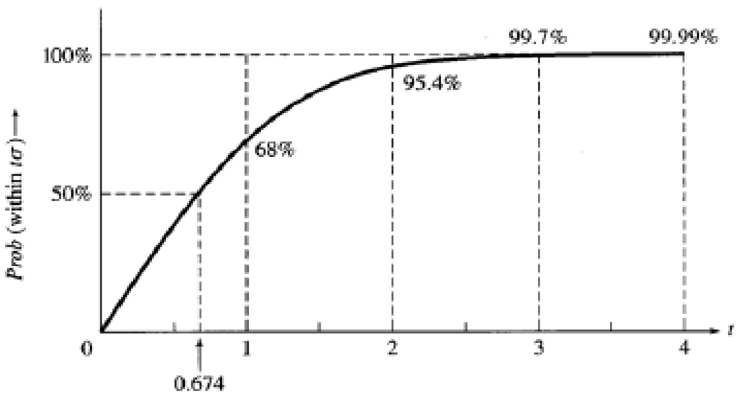
- This is the number of standard deviations that x_{best} differs from x_{exp}.
- Therefore,the probability of obtaining an answer that differs from x_{exp} by t or more standard deviations is:

Prob(outside tσ) = 1-Prob(within tσ))

"Acceptability" of a measured result
Conventions

• Large probability means likely outcome and hence reasonable discrepancy.
• "reasonable" is a matter of convention…

• We define:



erf(t) – error function

< 5 % - significant discrepancy, $t > 1.96$

< 1 % - highly significant discrepancy, $t > 2.58$

boundary for unreasonable improbability

If the discrepancy is beyond the **chosen** boundary for unreasonable improbability, ==> the theory and the measurement are incompatible (at the stated level)

Useful concept for complicated formula

- Often the quickest method is to calculate with the extreme values
  - $q = q(x)$
  - $q_{max} = q(\overline{x} + \delta x)$
  - $q_{min} = q(\overline{x} - \delta x)$

  □ $\delta q = (q_{max} - q_{min})/2$ (3.39)

Principle of Maximum Likelihood

- Best estimates of X and σ from N measurements (x_1 - x_N) are those for which $\text{Prob}_{X,\sigma}(x_i)$ is a maximum

Clicker Question 8

Upon flipping a coin three times, what are the chances of three heads in a row?

(a) 1
(b) 0.5
(c) 0.25
(d) 0.125
(e) 0.0625

The Principle of Maximum Likelihood

Recall the probability density for measurements of some quantity x (distributed as a Gaussian with mean X and standard deviation σ)

$$P_{X,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

Normal distribution is one example of *P(x)*.

Now, lets make repeated measurements of x to help reduce our errors.

$$x_1, \; x_2, \; x_3, \; \ldots, \; x_n$$

We define the Likelihood as the product of the probabilities. The larger L, the more likely a set of measurements is.

$$L = P(x_1)P(x_2)P(x_3)\ldots P(x_n)$$

Is *L* a Probability?

Why does max *L* give the best estimate?

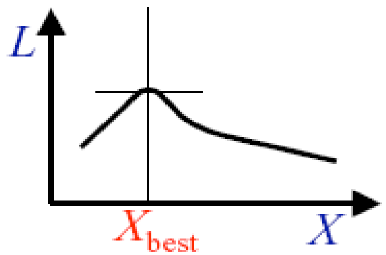
The best estimate for the parameters of *P(x)* are those that maximize *L*.

Yagil

Using the Principle of Maximum Likelihood:

Prove the mean is best estimate of X

Assume X is a parameter of $P(x)$.
When L is maximum, we must have: $\frac{\partial L}{\partial X} = 0$



Lets assume a Normal error distribution and find the formula for the best value for X.

$$L = P(x_1)P(x_2)\ldots P(x_n) = \prod_{i=1}^{n} P(x_i)$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - X)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\sum_{i=1}^{n} \frac{(x_i - X)^2}{2\sigma^2}}$$

$$L = Ce^{-\chi^2/2}$$

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - X)^2}{\sigma^2}$$ Defininition

$$\frac{\partial L}{\partial X} = 0 = Ce^{-\chi^2/2} \frac{-1}{2} \frac{\partial \chi^2}{\partial X}$$

$$\Rightarrow \quad \frac{\partial \chi^2}{\partial X} = 0$$

$$\frac{\partial \chi^2}{\partial X} = \frac{1}{\sigma^2} \sum_{i=1}^{n} 2(x_i - X) = 0$$

$$\sum_{i=1}^{n} (x_i - X) = 0$$

$$\sum_{i=1}^{n} x_i - nX = 0$$

$$X = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Q.E.D.
the mean

What is the Error on the Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Formula for mean of measurements. (We just proved that this is the best estimate of the true x.)

Now, use propagation of errors to get the error on the mean.

$$\sigma_{\bar{x}} = \frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1} \oplus \frac{\partial \bar{x}}{\partial x_2}\sigma_{x_2} \oplus \ldots \oplus \frac{\partial \bar{x}}{\partial x_n}\sigma_{x_n}$$

$$\frac{\partial \bar{x}}{\partial x_i} = \frac{1}{n}$$

$$\sigma_{\bar{x}} = \sqrt{\sum_{i=1}^{n}\left(\frac{\sigma_{x_i}}{n}\right)^2} = \sqrt{n\left(\frac{\sigma}{n}\right)^2} = \frac{\sigma}{\sqrt{n}}$$

What would you do if the x_i had different errors?

We got the error on the mean (SDOM) by propagating errors.

Yagil

The Four Experiments

- **Determine the average density of the earth**

Weigh the Earth, Measure its volume

– Measure simple things like lengths and times

– Learn to estimate and propagate errors

- **Non-Destructive measurements of densities, inner structure of objects**

– Absolute measurements *vs.* Measurements of variability

– Measure moments of inertia

– Use repeated measurements to reduce random errors

- **Construct and tune a shock absorber**

– Adjust performance of a mechanical system

– Demonstrate critical damping of your shock absorber

- **Measure coulomb force and calibrate a voltmeter.**

– Reduce systematic errors in a precise measurement.

Rotational Kinematics

Linear Kinematics

$$v_f = v_i + a\Delta t$$

$$\Delta s = v_i\Delta t + \tfrac{1}{2}a(\Delta t)^2$$

$$v_f^2 = v_i^2 + 2a\Delta s$$

Rotational Kinematics

$$\omega_f = \omega_i + \alpha\Delta t$$

$$\Delta\theta = \omega_i\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

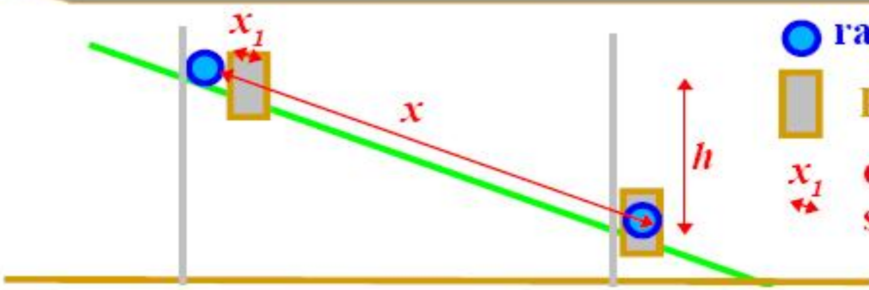
$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$s = \theta r \qquad v = \omega r \qquad a_t = \alpha r$$

Racquet Balls

We should check if the variation in *d* is much less than 10%.

Measuring I by Rolling Objects

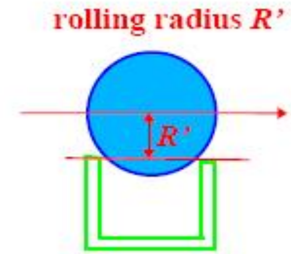


1. Measure M and R
2. Using photo gate timer measure the time, t, to travel distance x

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$ energy conservation

$$v = R'\omega$$ rolling radius

$$v = \frac{2x}{t}$$ for uniform acceleration



$$Mgh = \frac{1}{2}v^2\left(M + \frac{I}{R'^2}\right)$$

$$gh = \frac{2x^2}{t^2}\left(1 + \frac{I}{MR'^2}\right)$$

$$\frac{I}{MR'^2} = \left(\frac{ght^2}{2x^2} - 1\right)$$

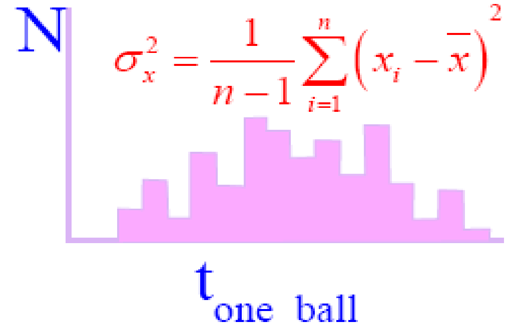
$$\tilde{I} \equiv \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right)$$

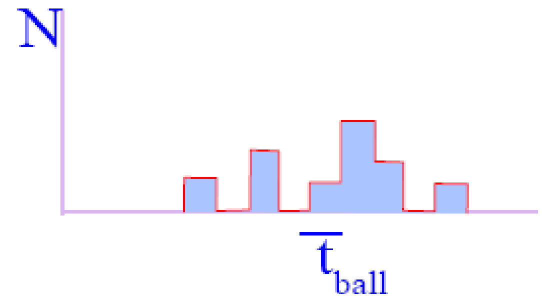
Measuring the Variation in Thickness of the Shell

- 1. Measure rolling time of one ball many times to determine the measurement error in *t*, $\sigma_{measurement}$
- 2. Measure rolling time of many balls to determine the total spread in *t*, σ_{total}
- 3. Calculate the spread in time due to ball manufacture, $\sigma_{manufacture}$, by subtracting the measurement error
- 4. Propagate error on *t* into error on *I* and then into error on thickness *d*

$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

statistical analysis





$$\sigma_{total} = \sigma_{manufacture} \oplus \sigma_{measruement}$$

$\sigma_t \rightarrow \sigma_I \rightarrow \sigma_d$

variation in *t* → variation in *I* → variation in *d*

Propagate Error from *I* to *d*

$$I = \frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}$$

$$z \equiv \frac{r}{R} \approx \frac{28.25 - 4.5 \text{ mm}}{28.25 \text{ mm}} \approx 0.841$$

← measured thickness and radius for one ball

d=4.5 mm R=28.25 mm

$d=R-r$

$$\tilde{I}(0.841) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571892$$

$$\tilde{I}(0.840) \equiv \frac{I}{MR^2} = \frac{2}{5}\frac{1-z^5}{1-z^3} \approx 0.571366$$

$$\frac{\partial z}{\partial \tilde{I}} = \frac{0.841 - 0.840}{0.571892 - 0.571366} = \frac{0.001}{0.00526} = 1.901$$

← $\delta z \longleftrightarrow \delta I$ numerically

$$\frac{\sigma_d}{d} = \frac{\sigma_r}{d} = \frac{R\sigma_z}{d} = \frac{R\tilde{I}}{d}\frac{\partial z}{\partial \tilde{I}}\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{(28.25 \text{ mm})(0.572)}{4.5 \text{ mm}}(1.901)\frac{\sigma_{\tilde{I}}}{\tilde{I}} = 6.826\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}}$$

Propagate Error from t to I

$$\tilde{I} = \frac{I}{MR^2} = \frac{R'^2}{R^2}\left(\frac{ght^2}{2x^2} - 1\right) \approx 0.572$$ from previous page

$$\frac{\partial \tilde{I}}{\partial t} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)$$ compute derivative

$$\sigma_{\tilde{I}} = \frac{R'^2}{R^2}\left(\frac{ght}{x^2}\right)\sigma_t$$ propagate error

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} = \frac{\left(\frac{ght}{x^2}\right)}{\left(\frac{ght^2}{2x^2} - 1\right)}\sigma_t \approx \frac{\left(\frac{ght}{x^2}\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t$$

work out fractional error numerically

$$\left(\frac{ght}{x^2}\right) = \frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)$$

$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx \frac{\frac{2}{t}\left(\frac{R^2}{R'^2}\tilde{I} + 1\right)}{\frac{R^2}{R'^2}(0.572)}\sigma_t = \frac{2\left(0.572 + \frac{R'^2}{R^2}\right)}{(0.572)}\frac{\sigma_t}{t} \approx 4\frac{\sigma_t}{t}$$

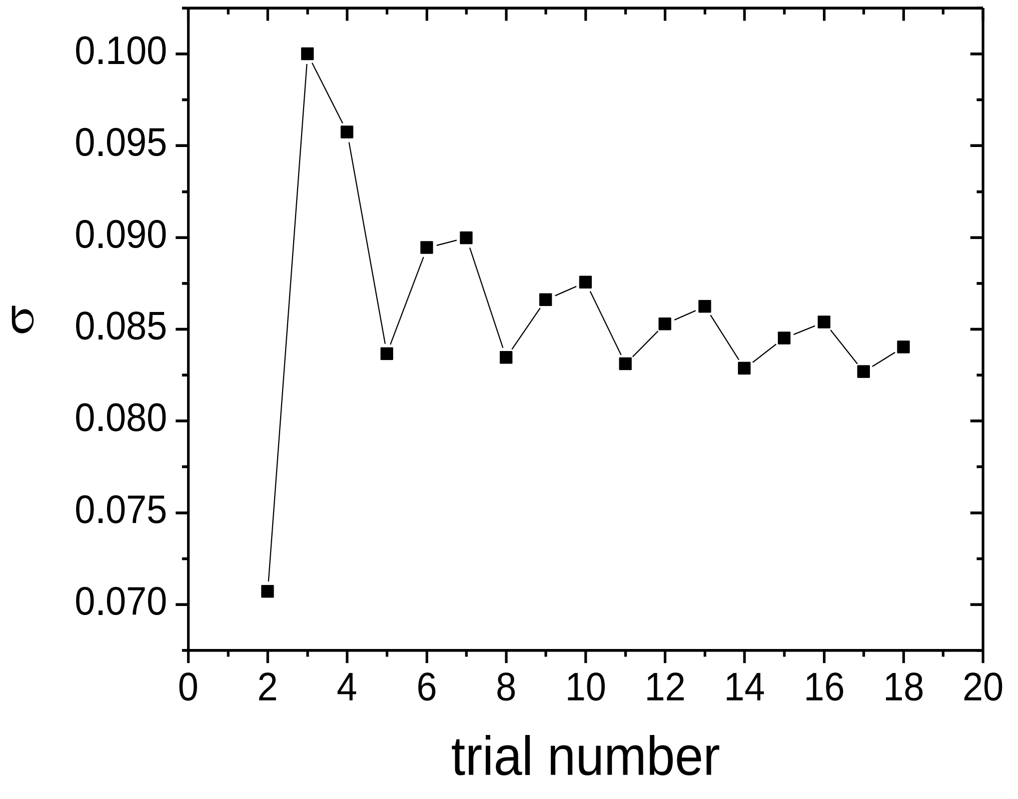
$$\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 4\frac{\sigma_t}{t}$$

$$\frac{\sigma_d}{d} \approx 6.8\frac{\sigma_{\tilde{I}}}{\tilde{I}} \approx 27\frac{\sigma_t}{t}$$

to get a 10% error on the thickness we need 0.37% error on the rolling time

accuracy can be improved by rolling each ball many times

Standard Deviation versus Trial Number



=STDEV(A$1:A2)

Remember

- Finish experiment #2
- Homework Taylor #8.6, 8.10
- Read Taylor through Chapter 9