Experiment 1, Physics 2BL

Deduce the mean density of the Earth.

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Preparation

Before this experiment, we recommend you review
or familiarize yourself with the following:

– *Chapters 1-4 in Taylor*

– *Partial derivatives*

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1. PHYSICS

1.1. Derivation of mean density in terms of gravity and radius, $\rho(g, R_E)$

We will be determining the mean density of the Earth
by calculating the radius of the earth, R_E, and the acceleration due to gravity at sea level, g. The force of gravity
on a mass m at sea level is given by the following:

$$F = \frac{GM_E m}{R_E^2} = mg$$

Where $G = 6.673 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$. Next we solve for the
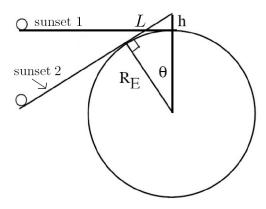mass of the Earth, M_E in terms of the acceleration due
to gravity at sea level, g:

$$M_E = \frac{gR_E^2}{G}$$

Which can be substituted into the expression for mean
density.

$$\boxed{\rho = \frac{M_E}{\frac{4}{3}\pi R_E^3} = \frac{3g}{4\pi G R_E}} \tag{1}$$

1.2. Measuring the radius of the Earth

The diagram above requires a little imagination. The
Sun is represented by the small circle that is making a
24 hour orbit (counterclockwise) around a large, non-rotating Earth. The tangent lines to the circle represent
the event of a sunset. The first sunset happens at the
bottom of the cliff, the second one at a height h above
sea level. L is the distance to the horizon from the height
of the cliff.

Questions

1. In our experiment, who will see the sunset first? (The
person on the top of the cliff or the person on the beach?)
If this experiment were to be conducted on the East
Coast, facing the Atlantic Ocean at sunrise, who would
see the sunrise first? Draw a diagram similar to the
one at the beginning of this section to explain the
East Coast situation.

We will now use geometry to solve for R_E given quantities that we can measure. The Pythagorean Theorem
gives us,

$$L^2 + R_E^2 = (R_E + h)^2$$

which can be rearranged and simplified for $h << R_E$:

$$L^2 = 2R_E h + h^2 \approx 2R_E h; \qquad L \approx \sqrt{2R_E h}$$

Now we set up a proportion to solve for the time in between the two sunsets, Δt. This corresponds to the time
it takes for the "sun" to go around the circumference of
the Earth:

$$\frac{L}{\Delta t} = \frac{2\pi R_E}{24hrs}$$

We can now solve for the R_E in terms of the height of
the cliff, h, and the time between sunsets, δt. We will
substitute $\omega = \frac{2\pi}{24hrs}$.

$$\boxed{R_E = \frac{2h}{\omega^2 (\Delta t)^2}} \tag{2}$$

Now lets re-derive this expression using the small angle aproximation approach. *Please check Appendix 1 at the end of this lab guide to refresh your memory about small angles.* This sort of argument is used very often in physics derivations.

Looking at the figure we get the following relationship:

$$\cos(\theta) = \frac{R_E}{R_E + h}$$

Using Taylor Expansion for $h/R_E << 1$ on both sides gives:

$$1 - \frac{\theta^2}{2} \approx 1 - \frac{h}{R_E}$$

Note that on the right hand side we used the expansion $\frac{1}{1+\epsilon} \approx 1 - \epsilon$ where ϵ is a small value. Rearranging and substituting $\theta = \omega \Delta t$ gives the same expression that we got in equation (2).

Questions

2. Using equation (2), estimate the difference in time of sunset between sea level and at 200m above sea level. Use $R_E \approx 6000km$ and $\omega = \frac{2\pi}{1 day}$.
(one day is $86,400s$).

3. For the previous example, calculate θ using the exact relationship for $\cos(\theta) = \frac{R_E}{R_E+h}$.
What is the value of θ^2 (terms that we are keeping)?
What is the value of θ^4 (terms that we are neglecting)?

1.3. Correction factors to R_E

In order to account for the height of the person on the beach h_{person} (their eyes are not exactly tangent to the sea level of the Earth), we must replace h in equation (2) with the following formula.

$$h_{Correct} = (\sqrt{h_{Cliff}} - \sqrt{h_{person}})^2$$

where h_{person} is defined as the distance from sea level to the eye level of the person on the beach, $h_{Cliff} = h + h_{person}$, and h has now been redefined as $h = L\cos(\theta)$ (the height difference between a person on the beach and a person on the cliff). See diagrams in the lecture notes and in section 3 of these guidelines for clarification on how each of these heights are defined.

We must also account for our latitude $\lambda_L = 32.87^0$ and the tilt of the earth in relation to the sun due to time of year $\lambda_S = -23.4^0 \sin(\frac{2\pi d}{365})$. Here d is the number of days since the last equinox (September 22 or March 20) where the tilt of the Earth is directly perpendicular to the rays of light that hit the equator.

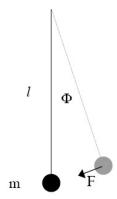$$C = \frac{1}{\cos^2(\lambda_L)\cos^2(\lambda_S) - \sin^2(\lambda_L)\sin^2(\lambda_S)}$$

If we account for these correction factors, the final equation for measuring R_E is the following:

$$\boxed{R_E = 2C(\frac{\sqrt{h_{Cliff}} - \sqrt{h_{person}}}{\omega \Delta t})^2} \tag{3}$$

Questions

4. Using $h_{Cliff} = 200m$ and $h_{person} = 2m$, calculate $h_{Correct}$. Suppose the observer on the beach lied about his height and is actually $1.5m$ tall. What would be the percent error $[\frac{calculated - actual}{actual} \times 100\%]$ on $h_{Correct}$ from using 2m instead of 1.5?

1.4. A Simple Pendulum

In order to measure g a simple pendulum will be constructed which will consist of a mass hanging from string. For this derivation a dot means first derivative in time and a double dot means second derivative in time. Initially the mass is displaced a small angle, $\phi(t = 0) = \phi_0$, with no initial angular velocity, $\dot{\phi}(t = 0) = 0$. A small component of the force of gravity torques the mass towards the equilibrium point. Using the small angle approximation (*See Appendix 1*):

$$F = -mg\sin(\phi) \approx -mg\phi$$

The linear acceleration is simply the length of the string times the angular acceleration:

$$F = ma = ml\alpha = ml\ddot{\phi}$$

Combining the previous two equations gives the following differential equation.

$$\ddot{\phi} + \frac{g}{l}\phi = 0$$

The characteristic equation $r^2 + \frac{g}{l} = 0$ has the roots $r = \pm i\sqrt{\frac{g}{l}} = i\omega$ and the solution is simple periodic motion:

$$\phi(t) = c_0 \cos(\omega t + \theta_0) = \phi_0 \cos(\omega t)$$

Where plugging in the initial conditions and solving for the constants gives $c_0 = \phi_0$ and $\theta_0 = 0$. The useful part of this derivation is the period of these oscillations,

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}} \qquad (4)$$

Questions

5. If you were constructing a pendulum what things would you take into consideration to simulate an idealized small-angle simple pendulum? Specifically what should the mass of the string be in comparison to that of the mass? What about the length of the string or the shape of the mass? What would be the effect of conducting this experiment in a vacuum as opposed to air?

6. Rearrange equation (4) to solve for g in terms of l and T. Your lab partner measures the length of your string as 45.3 cm but has undershot it by 3 cm because he neglected to measure all the way to the center of mass. If your group measures the period to be 1.4 s what is the value of g that you calculate? What value should you have gotten?

2. METHODS FOR STATISTICAL ANALYSIS

The following are math tools that you will need to use in this experiment and the ones to come. For extra practice you will be directed to homework problems in Taylor. The solutions to these problems will be posted on the course website.

2.1. Significant Figures and Rounding

Whenever you finish measuring or calculating a value and its uncertainty you **MUST** round your answer to its proper form.

First off let's refresh your memory about **significant digits**. Non zero digits are always significant. A zero is only significant if it is in between significant digits (like in 4009) or if there is a decimal point and the zeros are to the right of significant digits (like in 63.00 (4 sig figs) but not in .00049(2 sig figs)).

There are two simple rules for rounding:

1. Round the error to one sig fig, unless it is a 1, in which case you leave two sig figs. For example if you calculate the error to be 0.000459 you must round it to 0.0005. If the error is 1.356 then you round it to 1.4.

2. Round your value so that it ends with the same digits place as the error. For example 9.874501 $\pm$ 0.000459 gets rounded to 9.8745 $\pm$ 0.0005. Another example is 54590 $\pm$ 2349 which gets rounded to 55000 $\pm$ 2000.

Perhaps students get the feeling that if they round off their final answer to a few digits it looks like they didn't do as much work. This is not the case. Always report a measurement and finish up a calculation by leaving an answer in properly rounded form along with its units!

Questions

7. Round the following values:
(a) $1.375 \times 10^3 \pm 58$ m
(b) 0.035015 ± 0.000126 g

For extra practice see Taylor #3.10

See Appendix 3 for additional rules on sig figs and rounding

2.2. Random Errors

Many measurements are not particularly easy to make and have a different value for each trial. This is because the measurement device is more precise than the statistical fluctuations of the measurement. An example would be measuring the period of a pendulum. The stopwatch may be able to give the time to within 0.1s, but the period you measure varies by up to 0.8s. Suppose you make N measurements of a value for x and distinguish them with the subscript i. In order to get an average value for x you would sum over the different values x_i and divide by the total number of measurements.

$$\bar{x} = \frac{\sum x_i}{N}$$

In order to get the standard deviation, which would be the error on x, you must subtract each value x_i from the mean, square those deviations, sum over them, divide by $N-1$, and then take the square root.

$$\sigma_x = \sqrt{\frac{\sum(\bar{x}-x_i)^2}{N-1}}$$

A handy way of keeping track of this process is with the following table. The idea is to fill out the first column, use the mean to fill out the second column, etc. The middle column is not completely necessary, though sometimes it is useful to compare the different deviations to see whether any of the measurements are suspiciously off.

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
x_1	...	...
x_2	...	...
...	...	...
x_N	...	...
$\sum x_i$		$\sum (\bar{x} - x_i)^2$
$\bar{x}$		σ_x

For extra practice see Taylor #4.6

The nice thing about random errors is that they can be reduced by simply taking more trials. With every new measurement you are increasing N and decreasing the standard deviation, σ_x.

2.3. Instrumentational Errors

Many measurements you make might be the same every time you try to measure them, such as the length of a string or the mass of a ball, however they are not known to infinite accuracy. Here are the rules for finding the error of a scientific instrument:

1. If the scale reads 32.1 g that means you know the mass as 32.1 ± 0.1 g.

2. If your ruler has tick marks for every mm and you read 12.60 cm (making a rough estimate of the last digit) that means you know the length to 12.60 ± 0.05 cm.

The nice thing about some instrumentational errors is that they can be reduced by measuring a cluster of very similar things and then dividing by the number being measured. For example, measuring the width of a piece of paper by first measuring the thickness of 100 pages and then dividing the total length and the error on the total length by 100. Another example would be to measure the time for 10 periods of a pendulum to pass and then dividing the total time and the error on the total time by 10. This is very similar to the random errors that we found by recording the individual measurements and calculating the mean and standard deviation.

There are some instrumentational errors, however, that can not be reduced by repeated measurements. These usually involve a digital instrument that is being used to measure a value that does not fluctuate at the resolution of that instrument. An example would be a digital scale or a voltmeter.

2.4. Error Propagation

The following is the single most important equation you will need for this course. Suppose you are calculating a value f which is a function of x and y. You measure x to be $x_0 \pm \sigma_x$ and y to be $y_0 \pm \sigma_y$. The errors from the measured values must be added in quadrature, though each is "weighted" by its importance in the function f. The "weighting" comes from multiplying each error times its partial derivative evaluated at the values (x_0, y_0)

$$\sigma_f = \sqrt{[(\frac{\partial f}{\partial x})_{|(x_0, y_0)} \sigma_x]^2 + [(\frac{\partial f}{\partial y})_{|(x_0, y_0)} \sigma_y]^2} \quad (5)$$

Questions
8. Using Equation (5) write out the error propagating expression for σ_g when $g(l, T) = \frac{4\pi^2 l}{T^2}$

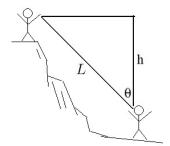*For extra practice see Taylor #3.28*

2.5. Systematic Errors

Sometimes even though you have accounted for all of the random fluctuations of your measurement, and you have noted the precision of your instruments, there might be some sort of hidden bias that is messing up your measurements. For example, you could spend all day trying to measure something with a ruler but if that ruler is not calibrated correctly then your measurements will be incorrect. Or let's say you are measuring the charge on a capacitor when you apply a voltage across it. Your measurement will be off unless you make sure to completely discharge the capacitor at the beginning. In general, systematic errors are associated with flawed equipment or a flawed experimental design. Unlike random errors, a systematic error can not be estimated by repeating a measurement many times. The focus of these experiments will be to try to account for all of the errors in your experiment. Sometimes the errors that you keep track of will not be sufficient for explaining why your calculated values differ from the expected values. It will then be your job to speculate what sorts of systematic errors could have lead you to incorrect results.

See section 4.1 in Taylor for a discussion of random Vs systematic errors.

3. EXPERIMENTAL PROCEDURE

3.1. Measuring The Height of The Cliff

At the beginning of first lab section we will walk to Black's beach. We will split up into groups of two: one

member of each group will go to the top of the cliffs and one member of each group will walk down to the beach. You will want to stand at a high position on the beach. Your TA will guide you to a good location. *You must remember where you stand because you will need to return to this same exact position when measuring the difference in time of the sunset.* The people on the beach will hold a large reflective mylar sheet up toward the people on the cliff. Then, the people on the cliff will use a rangefinder to measure the distance from the top to the bottom, L (See the cartoon above). Don't forget to record an associated uncertainty with this measurement! Those standing on the beach should NOT look up toward the people on the cliff while rangefinder measurements are being made.
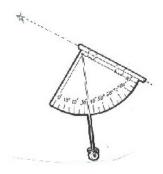
Once the students on the cliff have completed their rangefinder measurements, the people at the beach will measure the angle, θ to the top of the cliff using a hand-made sextant. This instrument allows you to use your eye to align the straight edge of a protractor and compare this angle to a vertical plumb line. Don't forget to record and associated uncertainty with this θ measurement. Keep in mind that a breeze might be blowing and the person holding the sextant might not be perfectly steady, so you will likely see the plum bob oscillate slightly. If the plum bob is oscillating through a couple degrees of what you record for θ, then you can not claim that your uncertainty in this measurement is less than a degree. Consider this when recording your uncertainty. *Note: the uncertainty is determined by the one making the measurement. It is not necessarily a fixed number. It depends on how well you believe you made your measurement*

After these measurements are made you will be able to calculate the difference in height between a person the on beach and a person on the cliff:

$$h = L\cos(\theta)$$

In order to use equation (2) to determine the radius of the Earth, you need h_{Cliff}, which depends on h and h_{person}. However, to calculate a more accurate value of R_E, you must use equation (3), which takes into account the fact that the eyes of the person on the beach are not at sea level when Δt (the time difference of sunset) measurement is made (see next section). See section 3.2

for details about measuring h_{person}. Also make sure you are using the correct angle and not $90^0 - \theta$.

This is a simple version of a sextant, an instrument used to measure angles in comparison to a plumb vertical line.

3.2. Difference in Time of Sunset

The individual group members will need to coordinate to return to the beach in pairs and stand at the same spots as before to record the time of sunset. Of course it will be necessary to synchronize watches before performing the measurement. Each person should measure the time at which he/she observes the sunset with an associated uncertainty. You might call these times $t_{Cliff} \pm \sigma_{t_{Cliff}}$ and $t_{beach} \pm \sigma_{t_{beach}}$. These measurements must be made on a clear day, otherwise the time difference of sunset is not easy to determine. If there are no clouds, the exact time of sunset should be defined as precisely when the last rays of light go below the horizon. From these measurements you can then determine $\Delta t = t_{Cliff} - t_{beach}$ and its uncertainty $\sigma_{\Delta t} = \sqrt{\sigma_{t_{Cliff}}^2 + \sigma_{t_{beach}}^2}$. After the time difference is determined it is possible to calculate R_E using equation (3).

Before the person on the beach leaves, he/she should estimate the height of his/her eye level from sea level, h_{person}. How you do this is up to you. Keep in mind, unless you have very precise surveying equipment, you estimation of h_{person} should have a large uncertainty. (Uncertainties on the order of centimeters would not be appropriate here.)

3.3. The Pendulum

For this part of the experiment you will be measuring the period of a pendulum as a function of angle. First choose a weight and measure its mass. Cut a length of string (fishing line recommended), tie it to the weight. In the lab there are large grey plastic disks with protractors attached to them. Use two stands to clamp one of these disks in a vertical position a couple feet from the table

top. Also in the lab there are silver colored pistons (with eye hooks) that fit snugly into the center hole of these plastic disks. Insert one of these pistons into the hole of your disk and use a nut, eye-hook, and black plastic washer to clamp the piston to the disk. For clarification, see the demo setup in lab. Do NOT use tape to secure the piston to the disk. Then tie the string to the eye hook screw. You should now have a pendulum setup that allows you to measure the angle of release. Measure the length of the string to the center of mass of the weight. *Note: The position of the center of mass is a rough estimate, so some guess should be made as to the error of this estimate.*

Now you will measure the period of the pendulum at different release angles using a stopwatch. Begin at some "large" angle, maybe 50 degrees, and release the weight. Measure the time for the pendulum to oscillate through p periods, where p should be at least 10. Repeat this measurement N times, where N should be at least 10 trials. Next, decrease the release angle by about 10 degrees and measure the time of p oscillations N times. Continue decreasing the release angle by these 10 degree increments, each time measuring the time of p oscillations N times, until you reach 10 degrees. Continue this same procedure but now decreasing by increments of 2 degrees. You should end with a measurement at the release angle of 2 degrees.

From this data you will then be able to make a graph of the period versus the release angle. And from this plot you will be able to decide which data is best to use to determine g and thus ρ. Keep in mind your data is NOT a measurement of a *single* period. For a given release angle θ_i, you will need to average the time data, then divide your average by p to find the average time of a *single* period, $\bar{T}_{\theta_i}$. See section 4.2 and 4.3 on how to determine the uncertainty on $\bar{T}_{\theta_i}$.

Questions

9. Under what conditions is equation (4) valid? Considering the procedure for the pendulum experiment (i.e. measuring the period at various angles), what values of T are valid in the equation for g (derived from equation (4)).

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4. ERROR ANALYSIS

The following is just a rough description of what you need to include in your report. For further details, refer to experiment 1 grading rubric posted on the website.

4.1. Instrumentational errors

Determine the errors for Δt, θ, L, and l. Justify how these errors were determined.

4.2. Random errors

At this point, for a given angle θ_i, you should have a set of time values $(t_1^p, t_2^p, t_3^p, ..., t_N^p)$. How do you approximate the uncertainty in these time measurements? If you think about it, you'll realize that the error in your time measurements classifies as random error, which implies you can quantify the uncertainty in each t^p measurement by calculating the standard deviation of your set of t^p values. In other words, the standard deviation of the set of t^p values corresponds to the uncertainty on each individual trial. For the first angle of release, calculate the standard deviation of the set of t^p values using the method outlined in section 2.2, then use Excel to compute the standard deviations for the time measurements at the remaining release angles. Keep in mind, this standard deviation, σ_{t^p} represents the uncertainty on the time of p oscillations. So, the uncertainty of a single period must be σ_{t^p} divided by the number of oscillations p (i.e. $\sigma_T = \frac{\sigma_{t^p}}{p}$). However, this uncertainty is not the uncertainty on the mean of a period. To find the uncertainty on the mean of a single period, you must calculate the standard deviation of the mean, which is discussed in the next section.

4.3. Standard Deviation of the Mean

In order to make a graph of period versus release angle that includes error bars and to calculate the uncertainties associated with R_E, g, and ρ, we need to know the uncertainty of the mean period for each release angle. The uncertainty in a mean can be calculated and is called the standard deviation of the mean. It is defined as $\sigma_{\bar{T}} = \frac{\sigma_T}{\sqrt{N}}$, where N is the number of trials. Now, calculate $\bar{T} \pm \sigma_{\bar{T}}$ for each angle of release. Use these mean values and uncertainties to make a plot of Period versus release angle that includes error bars.

Don't worry if you don't fully understand standard deviation of the mean. It will be used again and discussed in more detail in experiment 2. We bring it up here because it is needed in order to do the calculations correctly. At the least, you just need to be aware that the standard deviation and the standard deviation of the mean are different quantities. The former corresponds to the uncertainty on each individual measurement, whereas the latter corresponds to the uncertainty in the mean.

4.4. Summary of Pendulum Error Analysis (Sec 3.3, 4.2, 4.3)

- For release angle θ_i, you should have a set of time data $(t_1^p, t_2^p, t_3^p, ..., t_N^p)$.

- Calculate the average, $\bar{t}^p$, and the the standard deviation, σ_{t^p}, of this data.

- Divide $\bar{t}^p$ and σ_{t^p} by p to get average time of a *single* period, $\bar{T}$ and standard deviation of a single period σ_T.

- Calculate SDOM, $\sigma_{\bar{T}} = \frac{\sigma_T}{\sqrt{N}}$.

- Now you should have $\bar{T} \pm \sigma_{\bar{T}}$ for you data at θ_i.

- Repeat these calculations for data at each release angle.

4.5. Error Propagation

Use Equations (1), (3), and (4) to propagate errors to find σ_{R_E}, σ_g, and σ_ρ. Propagation the error for R_E is an extensive calculation. σ_{R_E} has already been worked out for you in the lecture slides and you may use the result if you like. To determine σ_g from equation (4), you will need to decide from your graph which $\bar{T} \pm \sigma_{\bar{T}}$ to use.

Useful note: $\theta \pm \sigma_\theta$ needs to be in radians when propagating error.

5. CONCLUSIONS

The expected values for this experiment are $R_E = 6,378km$, $g = 9.81\frac{m}{s^2}$, and $\rho = 5.52\frac{g}{cm^3}$ In your conclusion compare the values you calculated, along with their errors, to the the expected values. Do not simply report the percent error on each. If your final values are very different from the expected values, you must speculate as to the systematic errors that might have caused your answer to be off.

Questions

10. Suppose you calculated $\rho = 5.58 \pm .08\frac{g}{cm^3}$. Comparing this to the accepted value of ρ, what can you conclude about your determination of ρ? Why? What if you calculated $\rho = 5.61 \pm .08\frac{g}{cm^3}$; What could you conclude?
(Chapters 1-3 might be useful here.)

Appendix 1: The Small Angle Approximation

Sometimes it can be very useful to approximate a very small angle using the first few terms of the Taylor expansion for sine and cosine functions.

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}; \qquad \sin(\theta) \approx \theta$$

Which throws out terms that are of order θ^4 or smaller for cosine and terms that are of order θ^3 or smaller for sine.

Appendix 2: Lab Equipment- The Laser Range-finder

In order to measure the distance between the people on the beach and the people we will be using a laser rangefinder that measures the time delay between sending out a laser pulse and receiving it back after it bounces off of something. things that limit the accuracy are... this is very similar to the speedometer that a traffic cop uses, only that sends out multiple pulses to measure the Doppler shift of return pulses.

Appendix 3: Sig Fig and Rounding Addendum

In general, when you make a measurement, the uncertainty corresponding to this measurement should have one significant figure, unless that sig fig is a 1 (in which case you should have two sig figs). If your data is reported with the minimum allowable number of sig figs, then you might be wondering, "When do the rules of section 2.1 ever apply?" The rules of section 2.1 apply when you *calculate* some quantity using your data. For example, if we rearrange equation (4) we can find that $g = \frac{4\pi^2 l}{T^2}$. Suppose you measured $l = .315 \pm .005m$ and determined $\bar{T} \pm \sigma_{\bar{T}} = 1.132 \pm .008sec$ (See section 4.2 and 4.3 for details about $\bar{T}$ and $\sigma_{\bar{T}}$). Then $g = 9.7045.... \pm 0.8250....\frac{m}{s^2}$. However, when reporting your value, you must round according to the rules of section 2.1. In that case you would write down $g = 9.7 \pm 0.8\frac{m}{s^2}$ as the value you obtained for the gravitational constant at sea level.

What if you have subsequent calculations that require g and/or other quantities? *In such cases it is best not to round until you obtain your final result.* Most calculators can easily carry more digits than are likely to be significant. If you like, you may use a calculator to compute everything and thus eliminate the possibility of rounding error in your final result. Alternatively, if you would like to round during intermediate steps but also keep rounding error to a minimum, you can simply keep one extra sig fig in all quantities used to obtain your final result. For example, to determine ρ, you need g and R_E. Then, in the calculation for ρ you would use $g = 9.70 \pm 0.83\frac{m}{s^2}$.

Note: While you might keep one extra sig fig in g here for your calculation of ρ, when you report your result for g and compare it to the accepted value, you should only keep 1 sig fig as stated earlier.