**PHYSICS 210A : STATISTICAL PHYSICS**
**HW ASSIGNMENT #7 SOLUTIONS**

**(1)** For each of the two cluster diagrams in Fig. 1, find the symmetry factor $s_\gamma$ and write an expression for the cluster integral $b_\gamma(T)$.

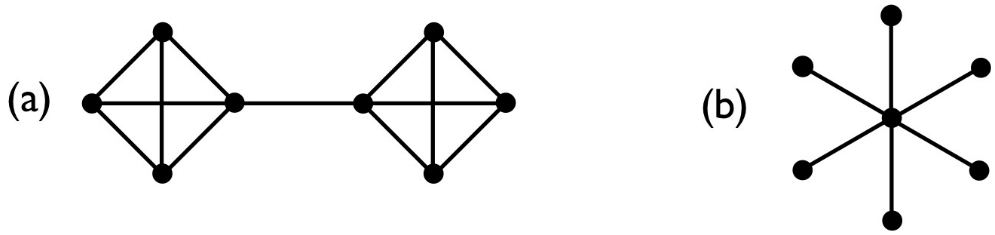


Figure 1: Mayer cluster expansion diagrams.

Solution :

The symmetry factors of the diagrams are $s_{\mathsf{a}} = 2 \cdot (3!)^2 = 72$ and $s_{\mathsf{b}} = 6! = 720$. To see this, note that sites 2, 3, and 4 and sites 5, 6, and 7 of figure 1a can be separately permuted in any of $3! = 6$ ways, and finally that the two triples themselves can be swapped to give a final factor of 2. For figure 1b, the sites $\{2, 3, 4, 5, 6, 7\}$ can be permuted in any way. One then has

$$b_{\mathsf{a}} = \frac{1}{72\,V} \int \prod_{i=1}^{8} d^d\!x_i \; f_{12}\, f_{13}\, f_{14}\, f_{23}\, f_{24}\, f_{34} \cdot f_{78}\, f_{68}\, f_{58}\, f_{67}\, f_{57}\, f_{56} \cdot f_{18}$$

$$b_{\mathsf{b}} = \frac{1}{720\,V} \int \prod_{i=1}^{7} d^d\!x_i \; f_{12}\, f_{13}\, f_{14}\, f_{15}\, f_{16}\, f_{17} \; .$$

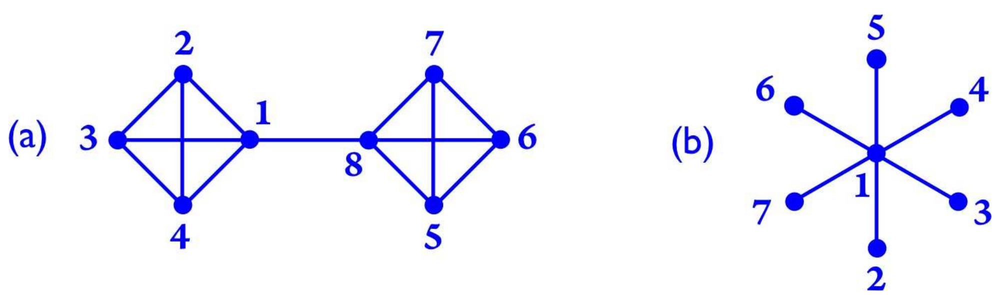


Figure 2: Labeled Mayer cluster expansion diagrams.

**(2)** Consider the one-dimensional Ising model with next-nearest neighbor interactions,

$$\hat{H} = -J \sum_n \sigma_n \sigma_{n+1} - K \sum_n \sigma_n \sigma_{n+2} \; ,$$

on a ring with $N$ sites, where $N$ is even. By considering consecutive pairs of sites, show that the partition function may be written in the form $Z = \mathsf{Tr}\,(R^{N/2})$, where $R$ is a $4 \times 4$

transfer matrix. Find $R$. *Hint:* It may be useful to think of the system as a railroad trestle, depicted in Fig. 2, with Hamiltonian

$$\hat{H} = -\sum_j \Big[ J\sigma_j\mu_j + J\mu_j\sigma_{j+1} + K\sigma_j\sigma_{j+1} + K\mu_j\mu_{j+1} \Big] \,.$$

Then $R = R_{(\sigma_j\mu_j),(\sigma_{j+1}\mu_{j+1})}$, with $(\sigma\mu)$ a composite index which takes one of four possible values $(++)$, $(+-)$, $(-+)$, or $(--)$.

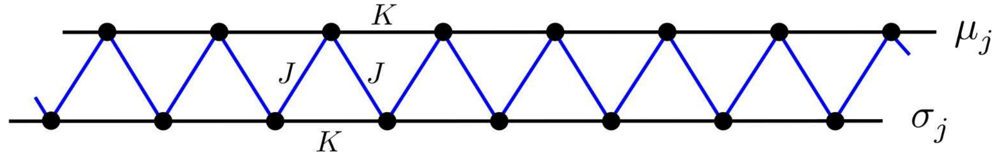


Figure 3: Railroad trestle representation of next-nearest neighbor chain.

Solution :

The transfer matrix can be read off from the Hamiltonian:

$$R_{(\sigma\mu),(\sigma'\mu')} = e^{\beta J\mu(\sigma+\sigma')}\, e^{\beta K(\sigma\sigma'+\mu\mu')} \,.$$

Expressed as a matrix of rank four, with rows and columns corresponding to $\{++, +-, -+, --\}$, we have

$$R = \begin{pmatrix} e^{2\beta(J+K)} & e^{2\beta J} & 1 & e^{-2\beta K} \\ e^{-2\beta J} & e^{-2\beta(J-K)} & e^{-2\beta K} & 1 \\ 1 & e^{-2\beta K} & e^{-2\beta(J-K)} & e^{-2\beta J} \\ e^{-2\beta K} & 1 & e^{2\beta J} & e^{2\beta(J+K)} \end{pmatrix} \,.$$

Querying WolframAlpha for the eigenvalues, we find

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{2}\Big[uv - (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v\Big] \\
\lambda_2 &= \tfrac{1}{2}\Big[uv + (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v\Big] \\
\lambda_3 &= \tfrac{1}{2}\Big[uv - (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v\Big] \\
\lambda_4 &= \tfrac{1}{2}\Big[uv + (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v\Big] \,,
\end{aligned}$$

where $u = e^{2\beta J}$ and $v = e^{2\beta K}$. The partition function on a ring of $N$ sites, with $N$ even, is

$$Z = \mathrm{Tr}\big(R^{N/2}\big) = \lambda_1^{N/2} + \lambda_2^{N/2} + \lambda_3^{N/2} + \lambda_4^{N/2} \,.$$