PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #5 SOLUTIONS

(1) For a noninteracting quantum system with single particle density of states $g(\varepsilon) = A\,\varepsilon^r$ (with $\varepsilon \geq 0$), find the first three virial coefficients for bosons and for fermions.

Solution :

We have

$$n(T,z) = \sum_{j=1}^{\infty} (\pm 1)^{j-1}\, C_j(T)\, z^j \qquad , \qquad p(T,z) = k_{\rm B}T \sum_{j=1}^{\infty} (\pm 1)^{j-1}\, z^j\, j^{-1}\, C_j(T)\, z^j \ ,$$

where

$$C_j(T) = \int\limits_{-\infty}^{\infty} \! d\varepsilon \; g(\varepsilon)\, e^{-j\varepsilon/k_{\rm B}T} = A\,\Gamma(r+1) \bigg(\frac{k_{\rm B}T}{j}\bigg)^{r+1} \ .$$

Thus, we have

$$\begin{aligned} \pm\, n v_T &= \sum_{j=1}^{\infty} j^{-(r+1)}\, (\pm z)^j \\ \pm\, p v_T / k_{\rm B}T &= \sum_{j=1}^{\infty} j^{-(r+2)}\, (\pm z)^j \ , \end{aligned}$$

where

$$v_T = \frac{1}{A\,\Gamma(r+1)\, (k_{\rm B}T)^{r+1}} \ .$$

has dimensions of volume. Thus, we let $x = \pm z$, and interrogate `Mathematica`:

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In[1]=  y = InverseSeries[x + x^2/2^(r+1) + x^3/3^(r+1) + x^4/4^(r+1) + O[x]^5]
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In[2]=  w = y + y^2/2^(r+2) + y^3/3^(r+2) + y^4/4^(r+2) + O[y]^5 .
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The result is

$$p = n k_{\rm B}T \Big[1 + B_2(T)\, n + B_3(T)\, n^2 + \ldots \Big] \ ,$$

where

$$\begin{aligned} B_2(T) &= \mp 2^{-2-r}\, v_T \\ B_3(T) &= \Big(2^{-2-2r} - 2 \cdot 3^{-2-r} \Big) v_T^2 \\ B_4(T) &= \pm 2^{-4-3r}\, 3^{-r} \Big(2^{3+2r} - 5 \cdot 3^r - 2^r\, 3^{1+r} \Big) v_T^3 \ . \end{aligned}$$

(2) How would you formulate the Lindemann melting criterion for Einstein phonons?

Solution :

For a one-dimensional harmonic oscillator, we have

$$\langle u^2 \rangle = \frac{\hbar}{2m\omega_0} \operatorname{ctnh}\big(\hbar\omega_0/2k_{\rm B}T\big) \ ,$$

where ω_0 is the oscillation frequency and m is the mass. For a d-dimensional Einstein solid, then, the Lindemann criterion should take the form

$$\langle \boldsymbol{u}^2 \rangle = \frac{d\hbar}{2m\omega_0} \operatorname{ctnh}\left(\hbar\omega_0/2k_{\rm B}T_{\rm L}\right) = (fa)^2 \ ,$$

where $f \approx \frac{1}{10}$, with a the lattice spacing. The Lindemann temperature is then

$$k_{\rm B}T_{\rm L} = \frac{\hbar\omega_0}{\ln\big(\frac{1+\eta}{1-\eta}\big)} \ ,$$

where

$$\eta = \frac{d\hbar}{2f^2 m\omega_0 a^2} \ .$$

Plugging in typical numbers, one finds $\eta \ll 1$ for most solids, assuming $\hbar\omega_0/k_{\rm B} \sim 100\,{\rm K}$. This procedure would then predict a melting temperature much higher than that observed for most solids.

(3) Derive the analogue of Stefan's Law for a two-dimensional blackbody. What happens if the photon dispersion is replaced by $\varepsilon(\boldsymbol{k}) = C|\boldsymbol{k}|^\alpha$?

Solution :

The power emitted per unit length of the boundary of such a two-dimensional blackbody is

$$\begin{aligned}
\frac{dP}{dL} &= \int \frac{d^2k}{(2\pi)^2}\ \hat{\boldsymbol{k}} \cdot \frac{\partial\varepsilon}{\partial\boldsymbol{k}} \cdot \frac{\varepsilon(\boldsymbol{k})}{e^{\varepsilon(\boldsymbol{k})/k_{\rm B}T} - 1}\, \Theta(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}) \\
&= \frac{\alpha C^2}{2\pi^2} \int\limits_0^\infty dk\, \frac{k^{2\alpha}}{e^{\beta C k^\alpha} - 1} \\
&= \frac{1}{2\pi^2}\, \Gamma(2+\alpha^{-1})\, \zeta(2+\alpha^{-1})\, C^{-\alpha^{-1}} (k_{\rm B}T)^{2+\alpha^{-1}} \\
&\equiv \sigma T^{2+\alpha^{-1}} \ .
\end{aligned}$$

Thus, for $\alpha = 1$, we have $P/L = \sigma T^3$.