PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #1

(1) Consider a system with K possible states $|\,i\,\rangle$, with $i \in \{1,\ldots,K\}$, where the transition rate W_{ij} between any two states is the same, with $W_{ij} = \gamma > 0$.

(a) Find the matrix Γ_{ij} governing the master equation $\dot{P}_i = -\Gamma_{ij}\, P_j$.

(b) Find all the eigenvalues and eigenvectors of Γ. What is the equilibrium distribution?

(c) Now suppose there are $2K$ possible states $|\,i\,\rangle$, with $i \in \{1,\ldots,2K\}$, and the transition rate matrix is

$$W_{ij} = \begin{cases} \alpha & \text{if} \quad (-1)^{ij} = +1 \\ \beta & \text{if} \quad (-1)^{ij} = -1\ , \end{cases}$$

with $\alpha, \beta > 0$. Repeat parts (a) and (b) for this system.

Solution :

(a) We have, from Eq. 3.3 of the Lecture Notes,

$$\Gamma_{ij} = \begin{cases} -W_{ij} = -\gamma & \text{if } i \neq j \\ \sum_k' W_{kj} = (K-1)\gamma & \text{if } i = j\ . \end{cases}$$

I.e. Γ is a symmetric $K \times K$ matrix with all off-diagonal entries $-\gamma$ and all diagonal entries $(K-1)\gamma$.

(b) It is convenient to define the unit vector $\vec{\psi} = K^{-1/2}(1,1,\ldots,1)$. Then

$$\Gamma = K\gamma\,\Big(\mathbb{I} - |\,\psi\,\rangle\langle\,\psi\,|\Big)\ .$$

We now see that $|\,\psi\,\rangle$ is an eigenvector of Γ with eigenvalue $\lambda = 0$, and furthermore that any vector orthogonal to $|\,\psi\,\rangle$ is an eigenvector of Γ with eigenvalue $K\gamma$. This means that there is a degenerate $(K-1)$-dimensional subspace associated with the eigenvalue $K\gamma$. The equilibrium distribution is given by $|\,P^{\text{eq}}\,\rangle = K^{-1/2}|\,\psi\,\rangle$, *i.e.* $P_i^{\text{eq}} = \frac{1}{K}$.

(c) Define the unit vectors

$$\vec{\psi}_{\mathsf{E}} = \tfrac{1}{\sqrt{K}}\,\big(0,1,0,\ldots,1\big)$$
$$\vec{\psi}_{\mathsf{O}} = \tfrac{1}{\sqrt{K}}\,\big(1,0,1,\ldots,0\big)\ .$$

Note that $\langle\,\psi_{\mathsf{E}}\,|\,\psi_{\mathsf{O}}\,\rangle = 0$. Furthermore, we may write Γ as

$$\Gamma = \tfrac{1}{2}K(3\alpha+\beta)\,\mathbb{I} + \tfrac{1}{2}K(\alpha-\beta)\,\mathbb{J} - K\alpha\,\Big(|\,\psi_{\mathsf{E}}\,\rangle\langle\,\psi_{\mathsf{E}}\,| + |\,\psi_{\mathsf{O}}\,\rangle\langle\,\psi_{\mathsf{E}}\,| + |\,\psi_{\mathsf{E}}\,\rangle\langle\,\psi_{\mathsf{O}}\,|\Big) - K\beta\,|\,\psi_{\mathsf{O}}\,\rangle\langle\,\psi_{\mathsf{O}}\,|$$

where $\mathbb{I}$ is the identity matrix and $\mathbb{J}_{nn'} = (-1)^n \, \delta_{nn'}$ is a diagonal matrix with alternating -1 and $+1$ entries. Note that $\mathbb{J} \, | \, \psi_{\mathsf{O}} \, \rangle = -| \, \psi_{\mathsf{O}} \, \rangle$ and $\mathbb{J} \, | \, \psi_{\mathsf{E}} \, \rangle = +| \, \psi_{\mathsf{E}} \, \rangle$. The key to deriving the above relation is to notice that

$$
\begin{aligned}
\mathbb{M} &= K\alpha \Big(| \, \psi_{\mathsf{E}} \, \rangle \langle \, \psi_{\mathsf{E}} \, | + | \, \psi_{\mathsf{O}} \, \rangle \langle \, \psi_{\mathsf{E}} \, | + | \, \psi_{\mathsf{E}} \, \rangle \langle \, \psi_{\mathsf{O}} \, | \Big) + K\beta \, | \, \psi_{\mathsf{O}} \, \rangle \langle \, \psi_{\mathsf{O}} \, | \\
&= \begin{pmatrix} \beta & \alpha & \beta & \alpha & \cdots & \beta & \alpha \\ \alpha & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\ \beta & \alpha & \beta & \alpha & \cdots & \beta & \alpha \\ \alpha & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta & \alpha & \beta & \alpha & \cdots & \beta & \alpha \\ \alpha & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha \end{pmatrix} .
\end{aligned}
$$

Now $\mathbb{J}$ has K eigenvalues $+1$ and K eigenvalues -1. There is therefore a $(K-1)$-dimensional degenerate eigenspace of Γ with eigenvalue $2K\alpha$ and a $(K-1)$-dimensional degenerate subspace with eigenvalue $K(\alpha+\beta)$. These subspaces are mutually orthogonal as well as being orthogonal to the vectors $| \, \psi_{\mathsf{E}} \, \rangle$ and $| \, \psi_{\mathsf{O}} \, \rangle$. The remaining two-dimensional subspace spanned by these vectors yields the reduced matrix

$$
\Gamma_{\rm red} = \begin{pmatrix} \langle \, \psi_{\mathsf{E}} \, | \, \Gamma \, | \, \psi_{\mathsf{E}} \, \rangle & \langle \, \psi_{\mathsf{E}} \, | \, \Gamma \, | \, \psi_{\mathsf{O}} \, \rangle \\ \langle \, \psi_{\mathsf{O}} \, | \, \Gamma \, | \, \psi_{\mathsf{E}} \, \rangle & \langle \, \psi_{\mathsf{O}} \, | \, \Gamma \, | \, \psi_{\mathsf{O}} \, \rangle \end{pmatrix} = \begin{pmatrix} K\alpha & -K\alpha \\ -K\alpha & K\alpha \end{pmatrix} .
$$

The eigenvalues in this subspace are therefore 0 and $2K\alpha$. Thus, Γ has the following eigenvalues:

$$
\begin{aligned}
&\lambda = 0 && \text{(nondegenerate)} \\
&\lambda = K(\alpha+\beta) && \text{(degeneracy } K-1) \\
&\lambda = 2K\alpha && \text{(degeneracy } K) \, .
\end{aligned}
$$

(2) A six-sided die is loaded so that the probability to throw a six is twice that of throwing a one. Find the distribution $\{p_n\}$ consistent with maximum entropy, given this constraint.

Solution :

The constraint may be written as $2p_1 - p_6 = 0$. Thus, $X_n^1 = 2\delta_{n,1} - \delta_{n,6}$, and

$$
p_n = \begin{cases} C\, e^{-2\lambda} & \text{if } n = 1 \\ C & \text{if } n \in \{2,3,4,5\} \\ C\, e^{\lambda} & \text{if } n = 6 \, . \end{cases}
$$

We solve for the unknowns C and λ by enforcing the constraints:

$$
\begin{aligned}
C\, e^{-2\lambda} + 4C + C\, e^{\lambda} &= 1 \\
2C\, e^{-2\lambda} - C\, e^{\lambda} &= 0 \, .
\end{aligned}
$$

The second equation gives $e^{3\lambda} = 2$, or $\lambda = \frac{1}{3}\ln 2$. Plugging this in the normalization condition, we have

$$C = \frac{1}{4 + 2^{1/3} + 2^{-2/3}} = 0.16798\ldots \ .$$

We then have

$$\begin{aligned} p_1 &= C\, e^{-2\lambda} = 0.10695\ldots \\ p_2 = p_3 = p_4 = p_5 &= C = 0.16798\ldots \\ p_6 &= C\, e^{\lambda} = 0.21391\ldots \ . \end{aligned}$$

(3) Consider a three-state system with the following transition rates:

$$W_{12} = 0 \quad , \quad W_{21} = \gamma \quad , \quad W_{23} = 0 \quad , \quad W_{32} = 3\gamma \quad , \quad W_{13} = \gamma \quad , \quad W_{31} = \gamma \, .$$

(a) Find the matrix Γ such that $\dot{P}_i = -\Gamma_{ij} P_j$.

(b) Find the equilibrium distribution P_i^{eq}.

(c) Does this system satisfy detailed balance? Why or why not?

Solution :

(a) Following the prescription in Eq. 3.3 of the Lecture Notes, we have

$$\Gamma = \gamma \begin{pmatrix} 2 & 0 & -1 \\ -1 & 3 & 0 \\ -1 & -3 & 1 \end{pmatrix} \ .$$

(b) Note that summing on the row index yields $\sum_i \Gamma_{ij} = 0$ for any j, hence $(1, 1, 1)$ is a left eigenvector of Γ with eigenvalue zero. It is quite simple to find the corresponding right eigenvector. Writing $\vec{\psi}^{\,\text{t}} = (a, b, c)$, we obtain the equations $c = 2a$, $a = 3b$, and $a + 3b = c$, the solution of which, with $a + b + c = 1$ for normalization, is $a = \frac{3}{10}$, $b = \frac{1}{10}$, and $c = \frac{6}{10}$. Thus,

$$P^{\text{eq}} = \begin{pmatrix} 0.3 \\ 0.1 \\ 0.6 \end{pmatrix} \ .$$

(c) The equilibrium distribution does not satisfy detailed balance. Consider for example the ratio $P_1^{\text{eq}} / P_2^{\text{eq}} = 3$. According to detailed balance, this should be the same as W_{12}/W_{21}, which is zero for the given set of transition rates.

(4) The cumulative grade distributions of six 'old school' (no + or - distinctions) professors from various fields are given in the table below. For each case, compute the entropy of the grade distribution.

Solution :

We compute the probabilities p_n for $n \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{F}\}$ and then the statistical entropy of the distribution, $S = -\sum_n p_n \log_2 p_n$ in units of bits. The results are shown in the amended table below. The maximum possible entropy is $S = \log_2 5 \approx 2.3219$.

Professor	A p_A	B p_B	C p_C	D p_D	F p_F	N S
Landau	1149 0.2007	2192 0.3829	1545 0.2699	718 0.1254	121 0.0211	5725 1.999
Vermeer	8310 0.8527	1141 0.1171	231 0.0237	56 0.0057	7 0.0007	9745 0.7365
Keynes	3310 0.2633	4141 0.3294	3446 0.2741	1032 0.0821	642 0.0511	12571 2.062
Noether	1263 0.2648	1874 0.3929	988 0.2071	355 0.0744	290 0.0608	4770 2.032
Borges	4002 0.5690	2121 0.3015	745 0.1059	109 0.0155	57 0.0081	7034 1.477
Salk	3318 0.2878	3875 0.3361	2921 0.2534	1011 0.0877	404 0.0350	11529 2.025
Turing	2800 0.2605	3199 0.2977	2977 0.2770	1209 0.1125	562 0.0523	10747 2.116

(5) A generalized two-dimensional cat map can be defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & p \\ q & pq+1 \end{pmatrix}}^{M} \begin{pmatrix} x \\ y \end{pmatrix} \ \text{mod } \mathbb{Z}^2 \ ,$$

where p and q are integers. Here $x, y \in [0,1]$ are two real numbers on the unit interval, so $(x, y) \in \mathbb{T}^2$ lives on a two-dimensional torus. The inverse map is

$$M^{-1} = \begin{pmatrix} pq+1 & -p \\ -q & q \end{pmatrix} \ .$$

Note that det $M = 1$.

(a) Consider the action of this map on a pixelated image of size $(lK) \times (lK)$, where $l \sim 4 - 10$ and $K \sim 20 - 100$. Starting with an initial state in which all the pixels in the left half of the array are "on" and the others are all "off", iterate the image with the generalized cat map, and compute at each state the entropy $S = -\sum_{\boldsymbol{r}} p_{\boldsymbol{r}} \ln p_{\boldsymbol{r}}$, where the sum is over the K^2 different $l \times l$ subblocks, and $p_{\boldsymbol{r}}$ is the probability to find an "on" pixel in subblock $\boldsymbol{r}$. (Take $p = q = 1$ for convenience, though you might want to explore other values).

Now consider a three-dimensional generalization (Chen *et al.*, *Chaos, Solitons, and Fractals* **21**, 749 (2004)), with

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ \text{mod} \ \mathbb{Z}^3 ,$$

which is a discrete automorphism of $\mathbb{T}^3$, the three-dimensional torus. Again, we require that both M and M^{-1} have integer coefficients. This can be guaranteed by writing

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p_x \\ 0 & q_x & p_x q_x + 1 \end{pmatrix} \quad , \quad M_y = \begin{pmatrix} 1 & 0 & p_y \\ 0 & 1 & 0 \\ q_y & 0 & p_y q_y + 1 \end{pmatrix} \quad , \quad M_z = \begin{pmatrix} 1 & p_z & 0 \\ q_z & p_z q_z + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and taking $M = M_x M_y M_z$, reminiscent of how we build a general $\mathsf{O}(3)$ rotation from a product of three $\mathsf{O}(2)$ rotations about different axes.

(b) Find M and M^{-1} when $p_x = q_x = p_y = q_y = p_z = q_z = 1$.

(c) Repeat part (a) for this three-dimensional generalized cat map, computing the entropy by summing over the K^3 different $l \times l \times l$ subblocks.

(d) 100 quatloos extra credit if you find a way to show how a three dimensional object (a ball, say) evolves under this map. Is it Poincaré recurrent?

Solution :

(a) See Figs. 2 and 3.

(b) We have

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad , \quad M_x^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} .$$

$$M_y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad , \quad M_y^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$M_z = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad M_z^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

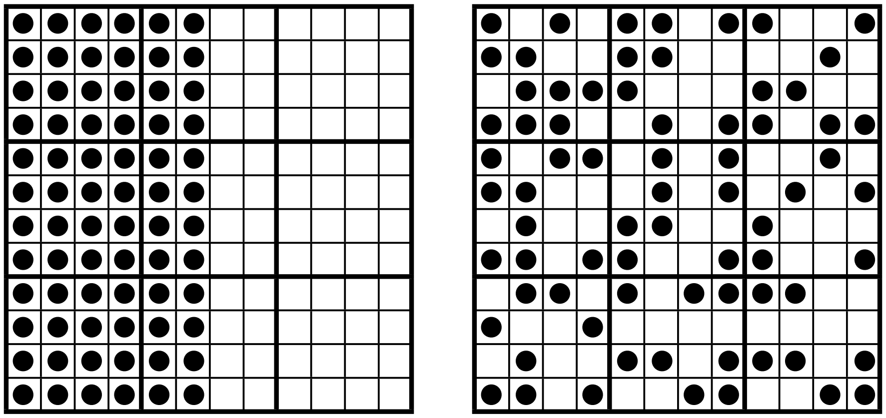

Figure 1: Two-dimensional cat map on a 12×12 square array with $l = 4$ and $K = 3$ shown. Left: initial conditions at $t = 0$. Right: possible conditions at some later time $t > 0$. Within each $l \times l$ cell $\boldsymbol{r}$, the occupation probability $p_{\boldsymbol{r}}$ is computed. The entropy $-p_{\boldsymbol{r}} \log_2 p_{\boldsymbol{r}}$ is then averaged over the K^2 cells.

Thus,

$$M = M_x M_y M_z = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 4 \end{pmatrix}$$

$$M^{-1} = M_z^{-1} M_y^{-1} M_x^{-1} = \begin{pmatrix} 4 & 0 & -1 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} .$$

Note that det $M = 1$.

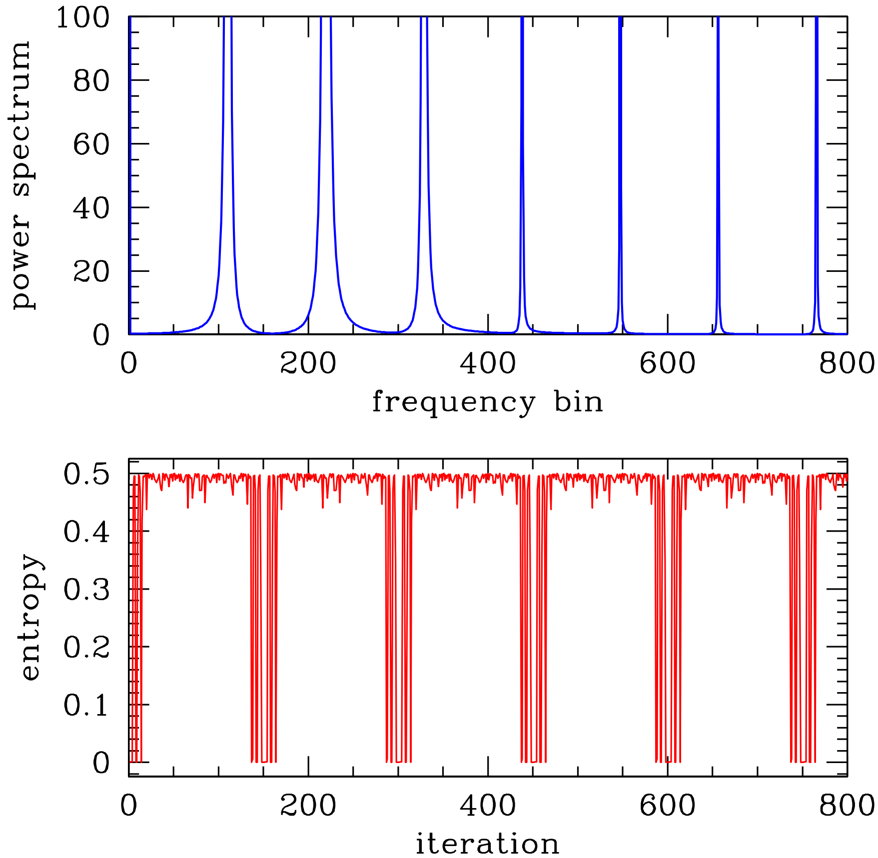

Figure 2: Coarse-grained entropy per unit volume for the iterated two-dimensional cat map ($p = q = 1$) on a 200×200 pixelated torus, with $l = 4$ and $K = 50$. Bottom panel: coarse-grained entropy per unit volume *versus* iteration number. Top panel: power spectrum of entropy *versus* frequency bin. A total of $2^{14} = 16384$ iterations were used.

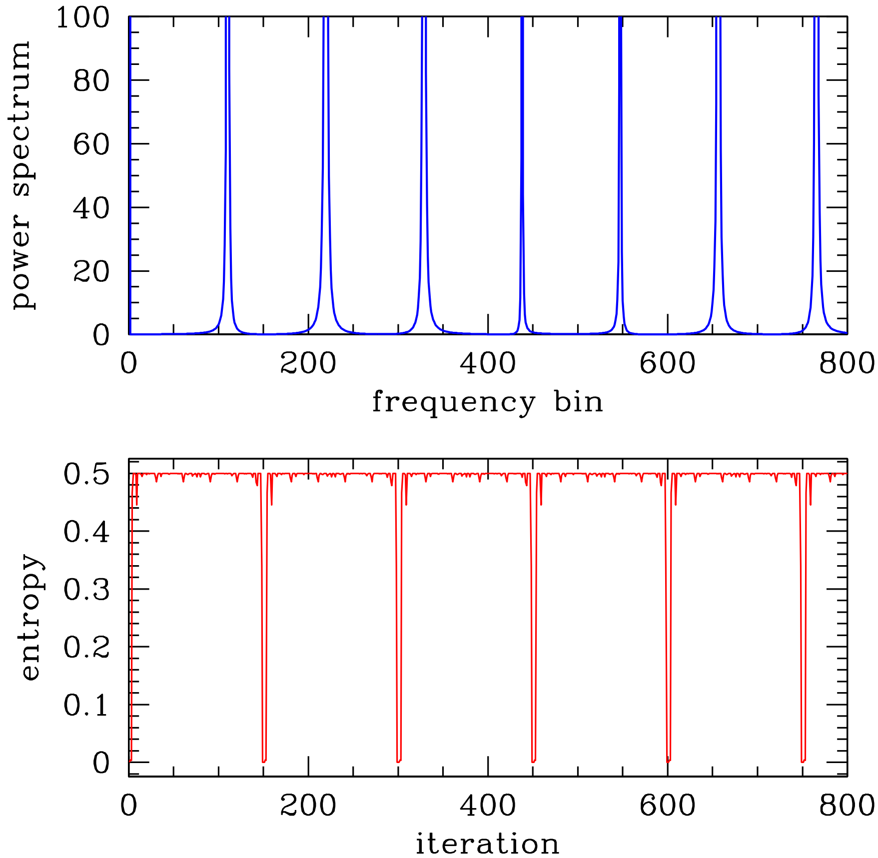

Figure 3: Coarse-grained entropy per unit volume for the iterated two-dimensional cat map ($p = q = 1$) on a 200×200 pixelated torus, with $l = 10$ and $K = 20$. Bottom panel: coarse-grained entropy per unit volume *versus* iteration number. Top panel: power spectrum of entropy *versus* frequency bin. A total of $2^{14} = 16384$ iterations were used.

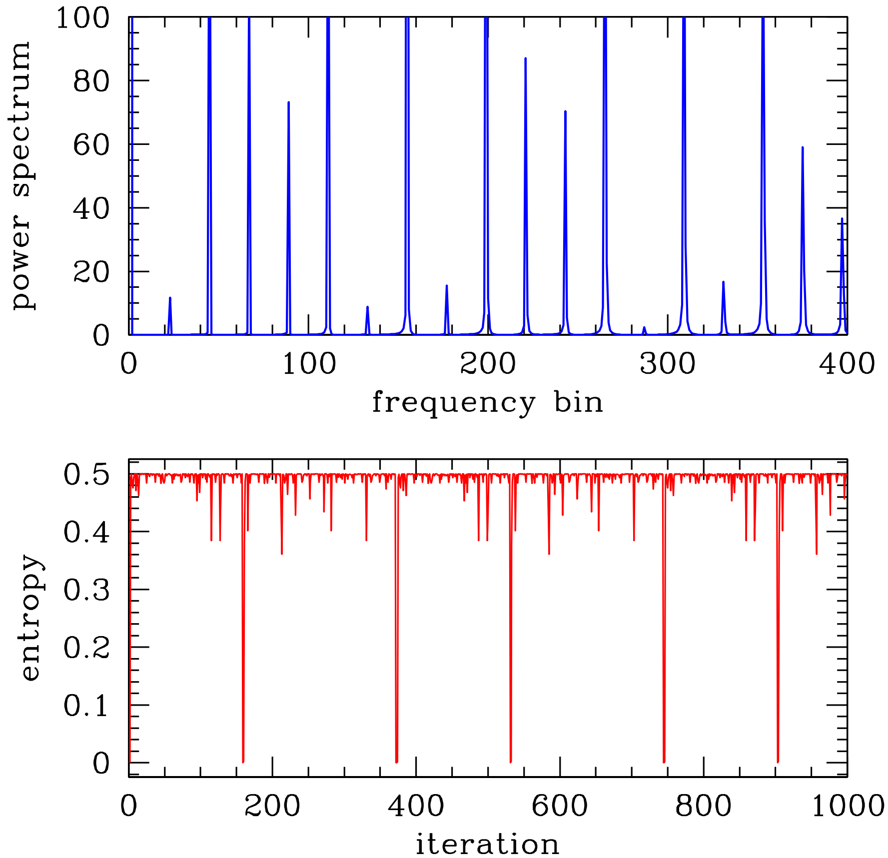

Figure 4: Coarse-grained entropy per unit volume for the iterated three-dimensional cat map ($p_x = q_x = p_y = q_y = p_z = q_z = 1$) on a $40 \times 40 \times 40$ pixelated three-dimensional torus, with $l = 4$ and $K = 10$. Bottom panel: coarse-grained entropy per unit volume *versus* iteration number. Top panel: power spectrum of entropy *versus* frequency bin. A total of $2^{14} = 16384$ iterations were used.