PHYSICS 210A : STATISTICAL PHYSICS
FINAL EXAMINATION

All parts are worth 5 points each

(1) `[40 points total]` Consider a noninteracting gas of bosons in d dimensions. Let the single particle dispersion be $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^\sigma$, where $\sigma > 0$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$. Show that $g(\varepsilon) = C\,\varepsilon^{p-1}\,\Theta(\varepsilon)$, and find C and p in terms of A, d, and σ. You may abbreviate the total solid angle in d dimensions as $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$.

(b) Under what conditions will there be a finite temperature $T_{\rm c}$ for Bose condensation?

(c) For $T > T_{\rm c}$, find an expression for the number density $n(T,z)$. You may find the following useful:

$$\int\limits_0^\infty \! d\varepsilon\, \frac{\varepsilon^{q-1}}{z^{-1}e^{\beta\varepsilon} - 1} = \Gamma(q)\,\beta^{-q}\,\mathsf{Li}_q(z)\ ,$$

where $\mathsf{Li}_q(z) = \sum_{j=1}^\infty z^j/j^q$ is the polylogarithm function. Note that $\mathsf{Li}_q(1) = \zeta(q)$.

(d) Assuming $T_{\rm c} > 0$, find an expression for $T_{\rm c}(n)$.

(e) For $T < T_{\rm c}$, find an expression for the condensate number density $n_0(T,n)$.

(f) For $T < T_{\rm c}$, compute the molar heat capacity at constant volume and particle number $c_{V,N}(T,n)$. Recall that $c_{V,N} = \frac{N_{\rm A}}{N}\big(\frac{\partial E}{\partial T}\big)_{V,N}$.

(g) For $T > T_{\rm c}$, compute the molar heat capacity at constant volume and particle number $c_{V,N}(T,z)$.

(h) Show that under certain conditions the heat capacity is discontinuous at $T_{\rm c}$, and evaluate $c_{V,N}(T_{\rm c}^\pm)$ just above and just below the transition.

(2) `[30 points total]` Consider the following model Hamiltonian,

$$\hat{H} = \sum_{\langle ij \rangle} E(\sigma_i, \sigma_j)\ ,$$

where each σ_i may take on one of three possible values, and

$$E(\sigma,\sigma') = \begin{pmatrix} -J & +J & 0 \\ +J & -J & 0 \\ 0 & 0 & +K \end{pmatrix}\ ,$$

with $J > 0$ and $K > 0$. Consider a variational density matrix $\varrho_{\rm v}(\sigma_1,\ldots,\sigma_N) = \prod_i \tilde\varrho(\sigma_i)$, where the normalized single site density matrix has diagonal elements

$$\tilde\varrho(\sigma) = \bigg(\frac{n+m}{2}\bigg)\delta_{\sigma,1} + \bigg(\frac{n-m}{2}\bigg)\delta_{\sigma,2} + (1-n)\,\delta_{\sigma,3}\ .$$

(a) What is the global symmetry group for this Hamiltonian?

(b) Evaluate $E = \mathsf{Tr}\,(\varrho_{\rm v}\,\hat{H})$.

(c) Evaluate $S = -k_{\rm B}\,\mathsf{Tr}\,(\varrho_{\rm v} \ln \varrho_{\rm v})$.

(d) Adimensionalize by writing $\theta = k_{\rm B}T/zJ$ and $c = K/J$, where z is the lattice coordination number. Find $f(n, m, \theta, c) = F/NzJ$.

(e) Find all the mean field equations.

(f) Find an equation for the critical temperature $\theta_{\rm c}$, and show graphically that it has a unique solution.

(3) `[30 points total]` Provide clear, accurate, and brief answers for each of the following:

(a) Explain what is meant by (i) recurrent, (ii) ergodic, and (iii) mixing phase flows.

(b) Why is it more accurate to compute response functions $\chi_{ij} = \partial m_i/\partial H_j$ rather than correlation functions $C_{ij} = \langle \sigma_i\, \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ in mean field theory? What is the exact thermodynamic relationship between χ_{ij} and C_{ij}?

(c) What is a tricritical point?

(d) Sketch what the radial distribution function $g(r)$ looks like for a simple fluid like liquid Argon. Identify any relevant length scales, as well as the proper limiting value for $g(r \to \infty)$.

(e) Discuss the First Law of Thermodynamics from the point of view of statistical mechanics.

(f) Explain what is meant by the Dulong-Petit limit of the heat capacity of a solid.