PHYSICS 140A : STATISTICAL PHYSICS
FINAL EXAMINATION SOLUTIONS
100 POINTS TOTAL

(1) Consider a system of N independent, distinguishable $S = 1$ objects, each described by the Hamiltonian

$$\hat{h} = \Delta\, S^2 - \mu_0 \mathsf{H}\, S \ ,$$

where $S \in \{-1, 0, 1\}$.

(a) Find $F(T, \mathsf{H}, N)$.
[10 points]

(b) Find the magnetization $M(T, \mathsf{H}, N)$. .
[5 points]

(c) Find the zero field susceptibility, $\chi(T) = \frac{1}{N}\frac{\partial M}{\partial \mathsf{H}}\Big|_{\mathsf{H}=0}$.
[5 points]

(d) Find the zero field entropy $S(T, \mathsf{H}=0, N)$. (*Hint : Take $H \to 0$ first.*)
[5 points]

Solution : The partition function is $Z = \zeta^N$, where ζ is the single particle partition function,

$$\zeta = \mathrm{Tr}\, e^{-\beta \hat{h}} = 1 + 2\, e^{-\Delta/k_\mathrm{B}T} \cosh\!\left(\frac{\mu_0 \mathsf{H}}{k_\mathrm{B}T}\right) \ . \tag{1}$$

Thus,

$$\text{(a)} \qquad \boxed{F = -N k_\mathrm{B} T \ln \zeta = -N k_\mathrm{B} T \, \ln\!\left[1 + 2\, e^{-\Delta/k_\mathrm{B}T} \cosh\!\left(\frac{\mu_0 \mathsf{H}}{k_\mathrm{B}T}\right)\right]} \tag{2}$$

The magnetization is

$$\text{(b)} \qquad \boxed{M = -\frac{\partial F}{\partial \mathsf{H}} = \frac{k_\mathrm{B}T}{Z} \cdot \frac{\partial Z}{\partial \mathsf{H}} = \frac{2\mu_0 \, \sinh\!\left(\frac{\mu_0 \mathsf{H}}{k_\mathrm{B}T}\right)}{e^{\Delta/k_\mathrm{B}T} + 2\cosh\!\left(\frac{\mu_0 \mathsf{H}}{k_\mathrm{B}T}\right)}} \tag{3}$$

To find the zero field susceptibility, we expand M to linear order in H, which entails expanding the numerator of M to first order in H and setting $\mathsf{H} = 0$ in the denominator. We then find

$$\text{(c)} \qquad \boxed{\chi(T) = \frac{2\mu_0^2}{k_\mathrm{B}T} \cdot \frac{1}{e^{\Delta/k_\mathrm{B}T} + 2}} \tag{4}$$

To find the entropy in zero field, it is convenient to set $\mathsf{H} \to 0$ first. The free energy is then given by $F(T, \mathsf{H}=0, N) = -N k_\mathrm{B} T \, \ln\left(1 + 2\, e^{-\Delta/k_\mathrm{B}T}\right)$, and therefore

$$\text{(d)} \qquad \boxed{S = -\frac{\partial F}{\partial T} = N k_\mathrm{B} \, \ln\left(1 + 2\, e^{-\Delta/k_\mathrm{B}T}\right) + \frac{N\Delta}{T} \cdot \frac{1}{2 + e^{\Delta/k_\mathrm{B}T}}} \tag{5}$$

(2) A classical gas consists of particles of two species: A and B. The dispersions for these species are

$$\varepsilon_{\rm A}(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} \quad , \quad \varepsilon_{\rm B}(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{4m} - \Delta \ .$$

In other words, $m_{\rm A} = m$ and $m_{\rm B} = 2m$, and there is an additional energy offset $-\Delta$ associated with the B species.

(a) Find the grand potential $\Omega(T, V, \mu_{\rm A}, \mu_{\rm B})$.
`[10 points]`

(b) Find the number densities $n_{\rm A}(T, \mu_{\rm A}, \mu_{\rm B})$ and $n_{\rm B}(T, \mu_{\rm A}, \mu_{\rm B})$.
`[5 points]`

(c) If $2{\rm A} \rightleftharpoons {\rm B}$ is an allowed reaction, what is the relation between $n_{\rm A}$ and $n_{\rm B}$? (*Hint : What is the relation between* $\mu_{\rm A}$ *and* $\mu_{\rm B}$*?*)
`[5 points]`

(d) Suppose initially that $n_{\rm A} = n$ and $n_{\rm B} = 0$. Find $n_{\rm A}$ in equilibrium, as a function of T and n and constants.
`[5 points]`

Solution : The grand partition function Ξ is a product of contributions from the A and B species, and the grand potential is a sum:

$$\text{(a)} \quad \boxed{\Omega = -V k_{\rm B} T \, \lambda_T^{-3} \, e^{\mu_{\rm A}/k_{\rm B}T} - 2^{3/2} \, V k_{\rm B} T \, \lambda_T^{-3} \, e^{(\mu_{\rm B}+\Delta)/k_{\rm B}T}} \tag{6}$$

Here, we have defined the thermal wavelength for the A species as $\lambda_T \equiv \lambda_{T,{\rm A}} = \sqrt{2\pi\hbar^2/mk_{\rm B}T}$. For the B species, since the mass is twice as great, we have $\lambda_{T,{\rm B}} = 2^{-1/2} \, \lambda_{T,{\rm A}}$.

The number densities are

$$n_{\rm A} = -\frac{1}{V} \cdot \frac{\partial \Omega}{\partial \mu_{\rm A}} = V \lambda_T^{-3} \, e^{\mu_{\rm A}/k_{\rm B}T} \tag{7}$$

$$n_{\rm B} = -\frac{1}{V} \cdot \frac{\partial \Omega}{\partial \mu_{\rm B}} = 2^{3/2} \, V \lambda_T^{-3} \, e^{(\mu_{\rm B}+\Delta)/k_{\rm B}T} \ . \tag{8}$$

If the reaction $2{\rm A} \rightleftharpoons {\rm B}$ is allowed, then the chemical potentials of the A and B species are related by $\mu_{\rm B} = 2\mu_{\rm A} \equiv 2\mu$. We then have

$$\text{(b)} \quad \boxed{n_{\rm A} \lambda_T^3 = e^{\mu/k_{\rm B}T}} \tag{9}$$

and

$$\text{(b)} \quad \boxed{n_{\rm B} \lambda_T^3 = 2^{3/2} \, e^{(2\mu+\Delta)/k_{\rm B}T}} \tag{10}$$

The relation we seek is therefore

$$\text{(c)} \qquad \boxed{n_{\rm B} = 2^{3/2}\,\big(n_{\rm A}\lambda_T^3\big)^2\, e^{\Delta/k_{\rm B}T}} \tag{11}$$

If we initially have $n_{\rm A} = n$ and $n_{\rm B} = 0$, then in general we must have

$$n_{\rm A} + 2n_{\rm B} = n \qquad \Longrightarrow \qquad n_{\rm B} = \tfrac{1}{2}\big(n - n_{\rm A}\big) \ . \tag{12}$$

Thus, eliminating $n_{\rm B}$, we have a quadratic equation,

$$2^{3/2}\,\lambda_T^3\, e^{\Delta/k_{\rm B}T}\, n_{\rm A}^2 = \tfrac{1}{2}(n - n_{\rm A}) \ , \tag{13}$$

the solution of which is

$$\text{(d)} \qquad \boxed{n_{\rm A} = \frac{-1 + \sqrt{1 + 16\sqrt{2}\, n\lambda_T^3\, e^{\Delta/k_{\rm B}T}}}{8\sqrt{2}\,\lambda_T^3\, e^{\Delta/k_{\rm B}T}}} \tag{14}$$

(3) A branch of excitations for a three-dimensional system has a dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{2/3}$. The excitations are bosonic and are not conserved; they therefore obey photon statistics.

(a) Find the single excitation density of states per unit volume, $g(\varepsilon)$. You may assume that there is no internal degeneracy for this excitation branch.
[10 points]

(b) Find the heat capacity $C_V(T,V)$.
[5 points]

(c) Find the ratio E/pV.
[5 points]

(d) If the particles are bosons with number conservation, find the critical temperature $T_{\rm c}$ for Bose-Einstein condensation.
[5 points]

Solution : We have, for three-dimensional systems,

$$g(\varepsilon) = \frac{1}{2\pi^2}\,\frac{k^2}{d\varepsilon/dk} = \frac{3}{4\pi^2 A}\, k^{7/3} \ . \tag{15}$$

Inverting the dispersion to give $k(\varepsilon) = (\varepsilon/A)^{3/2}$, we obtain

$$\text{(a)} \qquad \boxed{g(\varepsilon) = \frac{3}{4\pi^2}\,\frac{\varepsilon^{7/2}}{A^{9/2}}} \tag{16}$$

The energy is then

$$E = V \int_0^\infty d\varepsilon \; g(\varepsilon)\, \frac{\varepsilon}{e^{\varepsilon/k_B T} - 1}$$

$$= \frac{3V}{4\pi^2}\, \Gamma\big(\tfrac{11}{2}\big)\, \zeta\big(\tfrac{11}{2}\big)\, \frac{(k_B T)^{11/2}}{A^{9/2}} \; . \tag{17}$$

Thus,

$$\text{(b)} \quad \boxed{C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3V}{4\pi^2}\, \Gamma\big(\tfrac{13}{2}\big)\, \zeta\big(\tfrac{11}{2}\big)\, k_B \left(\frac{k_B T}{A}\right)^{9/2}} \tag{18}$$

The pressure is

$$p = -\frac{\Omega}{V} = -k_B T \int_0^\infty d\varepsilon \; g(\varepsilon)\; \ln\big(1 - e^{-\varepsilon/k_B T}\big) \tag{19}$$

$$= -k_B T \int_0^\infty d\varepsilon \; \frac{3}{4\pi^2}\, \frac{\varepsilon^{7/2}}{A^{9/2}}\; \ln\big(1 - e^{-\varepsilon/k_B T}\big)$$

$$= -\frac{3}{4\pi^2}\, \frac{(k_B T)^{11/2}}{A^{9/2}} \int_0^\infty ds\; s^{7/2}\; \ln\big(1 - e^{-s}\big)$$

$$= \frac{3V}{4\pi^2}\, \Gamma\big(\tfrac{9}{2}\big)\, \zeta\big(\tfrac{11}{2}\big)\, \frac{(k_B T)^{11/2}}{A^{9/2}} \; . \tag{20}$$

Thus,

$$\text{(c)} \quad \boxed{\frac{E}{pV} = \frac{\Gamma\big(\tfrac{11}{2}\big)}{\Gamma\big(\tfrac{9}{2}\big)} = \tfrac{9}{2}} \tag{21}$$

To find T_c for BEC, we set $z = 1$ ($i.e.$ $\mu = 0$) and $n_0 = 0$, and obtain

$$n = \int_0^\infty d\varepsilon \; g(\varepsilon)\, \frac{\varepsilon}{e^{\varepsilon/k_B T_c} - 1} \tag{22}$$

Substituting in our form for $g(\varepsilon)$, we obtain

$$n = \frac{3}{4\pi^2}\, \Gamma\big(\tfrac{9}{2}\big)\, \zeta\big(\tfrac{9}{2}\big) \left(\frac{k_B T}{A}\right)^{9/2} \; , \tag{23}$$

and therefore

$$\text{(d)} \quad \boxed{T_c = \frac{A}{k_B} \left(\frac{4\pi^2 n}{3\, \Gamma\big(\tfrac{9}{2}\big)\, \zeta\big(\tfrac{9}{2}\big)}\right)^{2/9}} \tag{24}$$

(4) Short answers:

(a) What are the conditions for a dynamical system to exhibit Poincaré recurrence? `[3 points]`
The time evolution of the dynamics must be invertible and volume-preserving on a phase space of finite total volume. For $\dot{\boldsymbol{\varphi}} = \boldsymbol{X}(\boldsymbol{\varphi})$ this requires that the phase space divergence vanish: $\boldsymbol{\nabla} \cdot \boldsymbol{X} = 0$.

(b) Describe what the term *ergodic* means in the context of a dynamical system. `[3 points]`
Ergodicity means that time averages may be replaced by phase space averages, *i.e.* $\big\langle f(\boldsymbol{\varphi})\big\rangle_T = \big\langle f(\boldsymbol{\varphi})\big\rangle_S$, where

$$\big\langle f(\boldsymbol{\varphi})\big\rangle_T = \lim_{T\to\infty} \frac{1}{T}\int\limits_0^T \! dt\, f\big(\boldsymbol{\varphi}(t)\big) \tag{25}$$

$$\big\langle f(\boldsymbol{\varphi})\big\rangle_S = \int\! d\mu\, \varrho(\boldsymbol{\varphi})\, f\big(\boldsymbol{\varphi}\big) \ , \tag{26}$$

where $\varrho(\boldsymbol{\varphi})$ is a phase space distribution. For the microcanonical ensemble,

$$\varrho(\boldsymbol{\varphi}) = \frac{\delta\big(E - H(\boldsymbol{\varphi})\big)}{\int\! d\mu\, \delta\big(E - H(\boldsymbol{\varphi})\big)} \ , \tag{27}$$

(c) What is the microcanonical ensemble? `[3 points]`
The microcanonical ensemble is defined by the phase space probability distribution $\varrho(\boldsymbol{\varphi}) = \delta\big(E - H(\boldsymbol{\varphi})\big)$, which says that all states that lie on the same constant energy hypersurface in phase space are equally likely.

(d) A system with $L = 6$ single particle levels contains $N = 3$ bosons. How many distinct many-body states are there? `[3 points]`
The general result for bosons is $\Omega_{\text{BE}}(L, N) = \binom{N+L-1}{N}$, so we have $\Omega = \binom{8}{3} = 56$.

(e) A system with $L = 6$ single particle levels contains $N = 3$ fermions. How many distinct many-body states are there? `[3 points]`
The general result for bosons is $\Omega_{\text{FD}}(L, N) = \binom{L}{N}$, so we have $\Omega = \binom{6}{3} = 20$.

(f) Explain how the Maxwell-Boltzmann limit results, starting from the expression for $\Omega_{\text{BE/FD}}(T, V, \mu)$. `[3 points]`
We have

$$\Omega_{\text{BE/FD}} = \pm k_{\text{B}} T \sum_\alpha \ln\big(1 \mp z\, e^{-\varepsilon_\alpha/k_{\text{B}}T}\big) \ . \tag{28}$$

The MB limit occurs when the product $z\, e^{-\varepsilon_\alpha/k_{\text{B}}T} \ll 1$, in which case

$$\Omega_{\text{BE/FD}} \longrightarrow \Omega_{\text{MB}} = -k_{\text{B}} T \sum_\alpha e^{(\mu - \varepsilon_\alpha)/k_{\text{B}}T} \ , \tag{29}$$

where the sum is over all energy eigenstates of the single particle Hamiltonian.

(g) For the Dieterici equation of state, $p\,(1-bn) = nk_BT\exp(-an/k_BT)$, find the second virial coefficient $B_2(T)$. `[3 points]`

We must expand in powers of the density n:

$$p = nk_BT\,\frac{e^{-an/k_BT}}{1-bn} = nk_BT\left(1 - \frac{an}{k_BT} + \ldots\right)(1 + bn + \ldots)$$
$$= nk_BT + \big(b\,k_BT - a\big)\,n^2 + \mathcal{O}(n^3)\ . \tag{30}$$

The virial expansion of the equation of state is

$$p = nk_BT\big(1 + B_2(T)\ + B_3(T)\,n^2 + \ldots\big)\ , \tag{31}$$

and so we identify

$$B_2(T) = b - \frac{a}{k_BT}\ . \tag{32}$$

(h) Explain the difference between the Einstein and Debye models for the specific heat of a solid. `[4 points]`

The Einstein model assumes a phonon density of states $g(\varepsilon) = \mathcal{C}_E\,\delta(\varepsilon - \varepsilon_0)$, while for the Debye model one has $g(\varepsilon) = \mathcal{C}_D\,\varepsilon^2\,\Theta(\varepsilon_D - \varepsilon)$, where $C_{E,D}$ are constants, and ε_D is a cutoff known as the Debye energy. At high temperatures, both models yield a Dulong-Petit heat capacity of $3Nk_B$, where N is the number of atoms in the solid. At low temperatures, however, the Einstein model yields an exponentially suppressed specific heat, while the specific heat of the Debye model obeys a T^3 power law.

(i) Who composed the song *yerushalayim shel zahav*? `[50 quatloos extra credit]`

The song was composed by Naomi Shemer in 1967. In 2005, it was revealed that it was based in part on a Basque folk song.