PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #6 SOLUTIONS

(1) A substance obeys the thermodynamic relation $E = aS^4/VN^2$.

(a) Compute the heat capacity $C_{V,N}$ in terms of N, V, and T.

(b) Compute the equation of state relating p, V, N, and T.

(c) Compute the ratio $C_{\varphi,N}/C_{V,N}$, where $C_{\varphi,N}$ is the heat capacity at constant φ and N, with $\varphi = V^2/T$.

Solution :

(a) We have

$$T = \frac{\partial E}{\partial S}\bigg|_{V,N} = \frac{4aS^3}{VN^2} \qquad \Rightarrow \qquad S = \left(\frac{TVN^2}{4a}\right)^{1/3} .$$

Plugging this into the expression for $E(S, V, N)$, we obtain

$$E(T, V, N) = \tfrac{1}{4}(4a)^{-1/3}\, T^{4/3} V^{1/3} N^{2/3} ,$$

and hence

$$C_{V,N} = \frac{\partial E}{\partial T}\bigg|_{V,N} = \tfrac{1}{3}(4a)^{-1/3}\, T^{1/3} V^{1/3} N^{2/3} .$$

(b) We have $T(S, V, N)$ and so we must find $p(S, V, N)$ and then eliminate S. Thus,

$$p = -\frac{\partial E}{\partial V}\bigg|_{S,N} = \frac{aS^4}{V^2N^2} = \tfrac{1}{4}(4a)^{-1/3}\, T^{4/3} V^{-2/3} N^{2/3} .$$

Cubing this result eliminates the fractional powers, yielding the equation of state

$$256a\, p^3 V^2 = N^2 T^4 .$$

Note also that $E = pV$ and $C_{V,N} = 4pV/3T$.

(d) We have $dE = đQ - p\,dV$, so

$$đQ = dE + p\,dV = C_{V,N}\,dT + \left\{ \left(\frac{\partial E}{\partial V}\right)_{T,N} + p \right\} dV .$$

Now we need to compute $dV\big|_{\varphi,N}$. We write

$$d\varphi = -\frac{V^2}{T^2}\,dT + \frac{2V}{T}\,dV ,$$

hence

$$dV\big|_{\varphi,N} = \frac{V}{2T}\,dT .$$

Substituting this into our expression for $đQ$, we have

$$C_{\varphi,N} = C_{V,N} + \left\{ \left(\frac{\partial E}{\partial V}\right)_{T,N} + p \right\} \frac{V}{2T} \, .$$

It is now left to us to compute

$$\left(\frac{\partial E}{\partial V}\right)_{T,N} = \tfrac{1}{12}(4a)^{-1/3}\, T^{4/3} V^{-2/3} N^{2/3} = \tfrac{1}{3}\, p \, .$$

We then have

$$C_{\varphi,N} = C_{V,N} + \frac{2pV}{3T} = \tfrac{3}{2}\, C_{V,N} \, .$$

Thus,

$$\frac{C_{\varphi,N}}{C_{V,N}} = \tfrac{3}{2} \, .$$

(2) Consider an engine cycle which follows the thermodynamic path in Fig. 1. The work material is ν moles of a diatomic ideal gas. BC is an isobar ($dp = 0$), CA is an isochore ($dV = 0$), and along AB one has

$$p(V) = p_{\mathsf{B}} + (p_{\mathsf{A}} - p_{\mathsf{B}}) \cdot \sqrt{\frac{V_{\mathsf{B}} - V}{V_{\mathsf{B}} - V_{\mathsf{A}}}} \, .$$

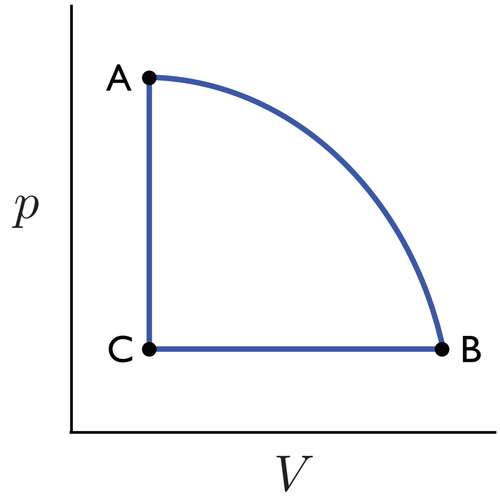

Figure 1: Thermodynamic path for problem 2.

(a) Find the heat acquired Q_{AB} and the work done W_{AB}.

(b) Find the heat acquired Q_{BC} and the work done W_{BC}.

(c) Find the heat acquired Q_{CA} and the work done W_{CA}.

(d) Find the work W done per cycle.

Solution :

Note that $p_{\mathsf{C}} = p_{\mathsf{B}}$ and $V_{\mathsf{C}} = V_{\mathsf{A}}$, so we will only need to use $\{p_{\mathsf{A}}, p_{\mathsf{B}}, V_{\mathsf{A}}, V_{\mathsf{B}}\}$ in our analysis. For a diatomic ideal gas, $E = \frac{5}{2}pV$.

(a) We first compute the work done along AB. Let's define u such that $V = V_{\mathsf{A}} + (V_{\mathsf{B}} - V_{\mathsf{A}})\,u$. Then along AB we have $p = p_{\mathsf{B}} + (p_{\mathsf{A}} - p_{\mathsf{B}})\sqrt{1-u}$, and

$$\begin{aligned} W_{\mathsf{AB}} &= \int\limits_{\mathsf{A}}^{\mathsf{B}} dV\, p \\ &= (V_{\mathsf{B}} - V_{\mathsf{A}}) \int\limits_0^1 du\, \Big\{ p_{\mathsf{B}} + (p_{\mathsf{A}} - p_{\mathsf{B}})\sqrt{1-u} \Big\} \\ &= p_{\mathsf{B}}(V_{\mathsf{B}} - V_{\mathsf{A}}) + \tfrac{2}{3}(V_{\mathsf{B}} - V_{\mathsf{A}})(p_{\mathsf{A}} - p_{\mathsf{B}}) \ . \end{aligned}$$

The change in energy along AB is

$$(\Delta E)_{\mathsf{AB}} = E_{\mathsf{B}} - E_{\mathsf{A}} = \tfrac{5}{2}(p_{\mathsf{B}} V_{\mathsf{B}} - p_{\mathsf{A}} V_{\mathsf{A}}) \ ,$$

hence

$$\begin{aligned} Q_{\mathsf{AB}} &= (\Delta E)_{\mathsf{AB}} + W_{\mathsf{AB}} \\ &= \tfrac{5}{6}\, p_{\mathsf{B}} V_{\mathsf{B}} - \tfrac{19}{6}\, p_{\mathsf{A}} V_{\mathsf{A}} + \tfrac{2}{3}\, p_{\mathsf{A}} V_{\mathsf{B}} + \tfrac{5}{3}\, p_{\mathsf{B}} V_{\mathsf{A}} \ . \end{aligned}$$

(b) Along BC we have

$$\begin{aligned} W_{\mathsf{BC}} &= p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}}) \\ (\Delta E)_{\mathsf{BC}} &= \tfrac{5}{2} p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}}) \\ Q_{\mathsf{BC}} &= (\Delta E)_{\mathsf{BC}} - W_{\mathsf{BC}} = \tfrac{3}{2} p_{\mathsf{B}}(V_{\mathsf{A}} - V_{\mathsf{B}}) \ . \end{aligned}$$

(c) Along CA we have

$$\begin{aligned} W_{\mathsf{BC}} &= 0 \\ (\Delta E)_{\mathsf{BC}} &= \tfrac{5}{2}(p_{\mathsf{A}} - p_{\mathsf{B}}) V_{\mathsf{A}} \\ Q_{\mathsf{CA}} &= (\Delta E)_{\mathsf{CA}} - W_{\mathsf{CA}} = \tfrac{5}{2}(p_{\mathsf{A}} - p_{\mathsf{B}}) V_{\mathsf{A}} \ . \end{aligned}$$

(c) The work done per cycle is

$$\begin{aligned} W &= W_{\mathsf{AB}} + W_{\mathsf{BC}} + W_{\mathsf{CA}} \\ &= \tfrac{2}{3}(V_{\mathsf{B}} - V_{\mathsf{A}})(p_{\mathsf{A}} - p_{\mathsf{B}}) \ . \end{aligned}$$

(3) For each of the following differentials, determine whether it is exact or inexact. If it is exact, find the function whose differential it represents.

(a) $xy^2\,dx + x^2y\,dy$

(b) $z\,dx + x\,dy + y\,dz$

(c) $x^{-2}\,dx - 2x^{-3}\,dy$

(d) $e^x\,dx + \ln(y)\,dy$

Solution :

We will represent each differential as $đA = \sum_\mu A_\mu\,dx^\mu$.

(a) $A_x = xy^2$ and $A_y = x^2y$, so $\frac{\partial A_x}{\partial y} = 2xy = \frac{\partial A_y}{\partial x}$. The differential is exact, and is dA, where $A(x,y) = \frac{1}{2}x^2y^2 + C$, where C is a constant.

(b) With $A_x = z$, $A_y = x$, and $A_z = y$, we have $\frac{\partial A_x}{\partial y} = 0 = \frac{\partial A_y}{\partial x}$, but $\frac{\partial A_x}{\partial z} = 1 \neq \frac{\partial A_z}{\partial x} = 0$. So the differential is inexact.

(c) $A_x = x^{-2}$ and $A_y = -2x^{-3}$, so $\frac{\partial A_x}{\partial y} = -2x^{-3}$ and $\frac{\partial A_y}{\partial x} = 0$, so the differential is inexact.

(d) $A_x = e^x$ and $A_y = \ln y$, so $\frac{\partial A_x}{\partial y} = 0 = \frac{\partial A_y}{\partial x} = 0$. The differential is exact, with $A(x,y) = e^x + y\ln y - y + C$.