PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #4 SOLUTIONS

(1) Consider a d-dimensional ultrarelativistic gas of classical indistinguishable particles with a dispersion $\varepsilon(\boldsymbol{p}) = c\,|\boldsymbol{p}|$.

(a) Find an expression for the grand potential $\Omega(T, V, \mu)$.

(b) Find the average number of particles $N(T, V, \mu)$.

(c) Find the entropy $S(T, V, \mu)$.

(d) Express the RMS fluctuations in the number of particle number, $(\Delta N)_{\rm RMS}$, in terms of the volume V, temperature T, and the pressure p.

Solution :

(a) The OCE partition function $Z(T, V, N)$ is computed in §4.2.4 of the Lecture Notes. One finds

$$Z(T, V, N) = \frac{V^N}{N!}\left(\frac{\Gamma(d)\,\Omega_d}{(\beta hc)^d}\right)^N .$$

From $\Xi = e^{-\beta\Omega} = \sum_{N=0}^{\infty} e^{\beta\mu N}\, Z(T, V, N)$, we obtain

$$\Omega(T, V, \mu) = -\frac{\Gamma(d)\Omega_d}{(hc)^d}\, V\,(k_{\rm B}T)^{d+1}\, e^{\mu/k_{\rm B}T} .$$

(b) The particle number is

$$N(T, V, \mu) = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = -(d+1)\frac{\Gamma(d)\,\Omega_d}{(hc)^d}\, V\,(k_{\rm B}T)^d\, e^{\mu/k_{\rm B}T}$$

(c) The entropy is

$$S(T, V, \mu) = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu} = \left((d+1)k_{\rm B} - \frac{\mu}{T}\right)\frac{\Gamma(d)\Omega_d}{(hc)^d}\, V\,(k_{\rm B}T)^d\, e^{\mu/k_{\rm B}T} .$$

(d) The variance of the number $\hat{N}$ is (see eqn. 4.138 of the Lecture Notes)

$$\mathsf{var}(\hat{N}) = k_{\rm B}T\left(\frac{\partial N}{\partial\mu}\right)_{T,V} = N = \frac{pV}{k_{\rm B}T} .$$

Thus,

$$(\Delta N)_{\rm RMS} = \sqrt{\mathsf{var}(\hat{N})} = \left(\frac{pV}{k_{\rm B}T}\right)^{1/2} .$$

(2) Consider again the d-dimensional classical ultrarelativistic gas with $\varepsilon(\boldsymbol{p}) = cp$.

(a) If $d = 3$, find the momentum distribution function $g(\boldsymbol{p})$.

(b) Again for $d = 3$, find a general formula for the moments $\langle\, |\boldsymbol{p}|^k \,\rangle$.

(c) Repeat parts (a) and (b) for the case $d = 2$.

(d) In $d = 3$, what is the distribution function $f(\boldsymbol{v})$ for velocities?

Solution :

(a) We have

$$g(\boldsymbol{p}) = \big\langle \delta(\boldsymbol{p} - \boldsymbol{p}_1) \big\rangle = \frac{e^{-\beta c p}}{\int d^3\!p \, e^{-\beta c p}} = \frac{c^3}{8\pi (k_{\rm B} T)^3} \, e^{-\beta c p} \ .$$

(b) The moments are

$$\big\langle |\boldsymbol{p}|^k \big\rangle = \tfrac{1}{2} (\beta c)^3 \int\limits_0^\infty \! dp \, p^{2+k} \, e^{-\beta c p} = \tfrac{1}{2} (k+2)! \, (\beta c)^{-k}$$

(c) In $d = 2$,

$$g(\boldsymbol{p}) = \big\langle \delta(\boldsymbol{p} - \boldsymbol{p}_1) \big\rangle = \frac{e^{-\beta c p}}{\int d^2\!p \, e^{-\beta c p}} = \frac{c^2}{2\pi (k_{\rm B} T)^2} \, e^{-\beta c p}$$

and

$$\big\langle |\boldsymbol{p}|^k \big\rangle = (\beta c)^2 \int\limits_0^\infty \! dp \, p^{1+k} \, e^{-\beta c p} = (k+1)! \, (\beta c)^{-k}$$

(d) The velocity is $\boldsymbol{v} = \frac{\partial \varepsilon}{\partial \boldsymbol{p}} = c \, \hat{\boldsymbol{p}}$. Thus, the magnitude is fixed at $|\boldsymbol{v}| = c$ and the direction is distributed isotropically, *i.e.*

$$f(\boldsymbol{v}) = \frac{\delta(v - c)}{4\pi c^2} \ .$$

(3) A classical gas of indistinguishable particles in three dimensions is described by the Hamiltonian

$$\hat{H} = \sum_{i=1}^N \Big\{ A \, |\boldsymbol{p}_i|^3 - \mu_0 H S_i \Big\} \ ,$$

where A is a constant, and where $S_i \in \{-1 \,,\, 0 \,,\, +1\}$ (*i.e.* there are three possible spin polarization states).

(a) Compute the free energy $F_{\mathsf{gas}}(T, H, V, N)$.

(b) Compute the magnetization density $m_{\mathsf{gas}} = M_{\mathsf{gas}}/V$ as a function of temperature, pressure, and magnetic field.

The gas is placed in thermal contact with a surface containing N_{s} adsorption sites, each with adsorption energy $-\Delta$. The surface is metallic and shields the adsorbed particles from the magnetic field, so the field at the surface may be approximated by $H = 0$.

(c) Find the Landau free energy for the surface, $\Omega_{\mathsf{surf}}(T, N_{\mathsf{s}}, \mu)$.

(d) Find the fraction $f_0(T, \mu)$ of empty adsorption sites.

(e) Find the gas pressure $p^*(T, H)$ at which $f_0 = \frac{1}{2}$.

Solution :

(a) The single particle partition function is

$$\zeta(T, V, H) = V \int \frac{d^3\!p}{h^3}\, e^{-Ap^3/k_\mathrm{B}T} \sum_{S=-1}^{1} e^{\mu_0 H S/k_\mathrm{B}T} = \frac{4\pi V k_\mathrm{B}T}{3Ah^3} \cdot \Big(1 + 2\cosh(\mu_0 H/k_\mathrm{B}T)\Big) \,.$$

The N-particle partition function is $Z_{\mathsf{gas}}(T, H, V, N) = \zeta^N/N!$, hence

$$F_{\mathsf{gas}} = -N k_\mathrm{B}T \left[\ln\!\left(\frac{4\pi V k_\mathrm{B}T}{3NAh^3} \right) + 1 \right] - N k_\mathrm{B}T \ln\Big(1 + 2\cosh(\mu_0 H/k_\mathrm{B}T)\Big)$$

(b) The magnetization density is

$$m_{\mathsf{gas}}(T, p, H) = -\frac{1}{V}\frac{\partial F}{\partial H} = \frac{p\mu_0}{k_\mathrm{B}T} \cdot \frac{2\sinh(\mu_0 H/k_\mathrm{B}T)}{1 + 2\cosh(\mu_0 H/k_\mathrm{B}T)}$$

We have used the ideal gas law, $pV = N k_\mathrm{B}T$ here.

(c) There are four possible states for an adsorption site: empty, or occupied by a particle with one of three possible spin polarizations. Thus, $\Xi_{\mathsf{surf}}(T, N_{\mathsf{s}}, \mu) = \xi^{N_{\mathsf{s}}}$, with

$$\xi(T, \mu) = 1 + 3\, e^{(\mu+\Delta)/k_\mathrm{B}T} \,.$$

Thus,

$$\Omega_{\mathsf{surf}}(T, N_{\mathsf{s}}, \mu) = -N_{\mathsf{s}} k_\mathrm{B}T \ln\Big(1 + 3\, e^{(\mu+\Delta)/k_\mathrm{B}T}\Big)$$

(d) The fraction of empty adsorption sites is $1/\xi$, *i.e.*

$$f_0(T, \mu) = \frac{1}{1 + 3\, e^{(\mu+\Delta)/k_\mathrm{B}T}}$$

(e) Setting $f_0 = \frac{1}{2}$, we obtain the equation $3\,e^{(\mu+\Delta)/k_\mathrm{B}T} = 1$, or

$$e^{\mu/k_\mathrm{B}T} = \tfrac{1}{3}\,e^{-\Delta/k_\mathrm{B}T} \ .$$

We now need the fugacity $z = e^{\mu/k_\mathrm{B}T}$ in terms of p, T, and H. To this end, we compute the Landau free energy of the gas,

$$\Omega_{\mathsf{gas}} = -pV = -k_\mathrm{B}T\,\zeta\,e^{\mu/k_\mathrm{B}T} \ .$$

Thus,

$$p^*(T,H) = \frac{k_\mathrm{B}T\,\zeta}{V}\,e^{\mu/k_\mathrm{B}T} = \frac{4\pi(k_\mathrm{B}T)^2}{9Ah^3}\cdot\Big(1 + 2\cosh(\mu_0 H/k_\mathrm{B}T)\Big)e^{-\Delta/k_\mathrm{B}T}$$