PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #3 SOLUTIONS

(1) Consider a system described by the Hamiltonian

$$\hat{H} = -H \sum_{i=1}^{N} \sigma_i + \Delta \sum_{i=1}^{N} (1 - \sigma_i^2) \, ,$$

where each $\sigma_i \in \{-1, 0, +1\}$.

(a) Compute the ordinary canonical partition function $Z(T, N, H, \Delta)$ and the free energy $F(T, N, H, \Delta)$.

(b) Find the magnetization $M(T, N, H, \Delta)$.

(c) Show that $\frac{\partial M}{\partial \Delta} = -\frac{\partial N_0}{\partial H}$, where $N_0 = \sum_{i=1}^{N} \delta_{\sigma_i,0}$.

Solution :

(a) We have

$$Z(T, V, N, \Delta) = \Big(e^{\beta H} + e^{-\beta\Delta} + e^{-\beta H} \Big)^N$$
$$F(T, V, N, \Delta) = -N k_{\rm B} T \ln\Big(2\cosh(\beta H) + e^{-\beta\Delta} \Big)$$

(b) The thermodynamic magnetization is given by

$$M = -\left(\frac{\partial F}{\partial H}\right)_{N,\Delta} = \frac{N \sinh(\beta H)}{\cosh(\beta H) + \frac{1}{2}\, e^{-\beta\Delta}} \, .$$

(c) The Hamiltonian can be written $\hat{H} = -HM + \Delta N_0$, since $\delta_{\sigma,0} = 1 - \sigma^2$ when $\sigma \in \{-1, 0, +1\}$. Thus,

$$N_0 = + \left(\frac{\partial F}{\partial \Delta}\right)_{H,N} \quad \Rightarrow \quad -\frac{\partial^2 F}{\partial H \, \partial \Delta} = \frac{\partial M}{\partial \Delta} = -\frac{\partial N_0}{\partial H} \, .$$

(2) Consider a three-dimensional gas of N identical particles of mass m, each of which has a magnetic dipole moment $\boldsymbol{m} = \mu_0 \hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is a three-dimensional unit vector. The Hamiltonian is

$$\hat{H} = \sum_{i=1}^{N} \left[\frac{\boldsymbol{p}_i^2}{2m} - \mu_0 \boldsymbol{H} \cdot \hat{\boldsymbol{n}}_i \right] .$$

(a) What is the grand potential $\Omega(T, V, \mu, \boldsymbol{H})$?

(b) Express $\boldsymbol{M}$ in terms of T, V, N, and $\boldsymbol{H}$, where N is the average number of particles.

(c) Find $M(T, V, N, \boldsymbol{H})$ to lowest order in the external field.

Solution :

(a) The contribution from the orientational ($\hat{\boldsymbol{n}}$) degrees of freedom to the single particle partition function is

$$\xi_{\hat{\boldsymbol{n}}} = \int \frac{d\hat{\boldsymbol{n}}}{4\pi}\, e^{\beta\mu_0 \boldsymbol{H}\cdot\hat{\boldsymbol{n}}} = \tfrac{1}{2}\int\limits_{-1}^{1} dx\, e^{\beta\mu_0 H x} = \frac{\sinh(\beta\mu_0 H)}{\beta\mu_0 H} \ ,$$

where the integral is done by choosing the $\hat{\boldsymbol{z}}$-axis to lie along $\boldsymbol{H}$, then integrating out over the azimuthal angle ϕ (yielding 2π), and finally over $x = \hat{\hat{H}} \cdot \hat{\boldsymbol{n}} = \cos\theta$. Then

$$\xi(T, H) = V\lambda_T^{-d}\, \xi_{\hat{\boldsymbol{n}}}(T, H)$$

and so $Z(T, V, \mu, \boldsymbol{H}) = \xi^N/N!$ is the canonical partition function. The grand potential, following the discussion in the Lecture Notes, is then

$$\Omega(T, V, \mu, \boldsymbol{H}) = -V k_{\rm B} T \lambda_T^{-d}\, e^{\mu/k_{\rm B}T} \cdot \frac{\sinh(\mu_0 H/k_{\rm B}T)}{\mu_0 H/k_{\rm B}T} \ .$$

(b) Let $\alpha \equiv \mu_0 H/k_{\rm B}T$, and define $\xi(\alpha) = \alpha^{-1}\sinh\alpha$. We have

$$\boldsymbol{M} = -\frac{\partial\Omega}{\partial\boldsymbol{H}} = V\lambda_T^{-d}\, e^{\mu/k_{\rm B}T} \cdot \mu_0 \hat{\boldsymbol{H}}\, \xi'(\alpha)\big|_{\alpha=\mu_0 H/k_{\rm B}T} \ .$$

The particle number is

$$N = -\frac{\partial\Omega}{\partial\mu} = V\lambda_T^{-d}\, e^{\mu/k_{\rm B}T}\, \xi(\alpha) \ ,$$

hence

$$\boldsymbol{M} = N\mu_0 \hat{\boldsymbol{H}} \cdot \frac{\xi'(\alpha)}{\xi(\alpha)} = N\mu_0 \hat{\boldsymbol{H}} \cdot \Big(\operatorname{ctnh}\alpha - \alpha^{-1}\Big) \ .$$

Note that we have here used

$$\frac{\partial H}{\partial \boldsymbol{H}} = \hat{\boldsymbol{H}} = \frac{\boldsymbol{H}}{H} \ ,$$

where $H = |\boldsymbol{H}|$.

(c) We expand $\operatorname{ctnh}\alpha = \frac{1}{\alpha} + \frac{\alpha}{3} + \mathcal{O}(\alpha^3)$ in a Laurent series. To lowest order in $\boldsymbol{H}$, then,

$$\boldsymbol{M} = \frac{N\mu_0^2\, \boldsymbol{H}}{3k_{\rm B}T} \ .$$