PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #1 SOLUTIONS

(1) The *information entropy* of a distribution $\{p_n\}$ is defined as $S = -\sum_n p_n \log_2 p_n$, where n ranges over all possible configurations of a given physical system and p_n is the probability of the state $|n\rangle$. If there are Ω possible states and each state is equally likely, then $S = \log_2 \Omega$, which is the usual dimensionless entropy in units of $\ln 2$.

Consider a normal deck of 52 distinct playing cards. A new deck always is prepared in the same order (A♠ 2♠ $\cdots$ K♣).

(a) What is the information entropy of the distribution of new decks?

(b) What is the information entropy of a distribution of completely randomized decks?

Now consider what it means to shuffle the cards. In an ideal *riffle shuffle*, the deck is split and divided into two equal halves of 26 cards each. One then chooses at random whether to take a card from either half, until one runs through all the cards and a new order is established (see figure).

Figure 1: The riffle shuffle.

(c) What is the increase in information entropy for a distribution of new decks that each have been shuffled once?

(d) Assuming each subsequent shuffle results in the same entropy increase (*i.e.* neglecting redundancies), how many shuffles are necessary in order to completely randomize a deck?

(e) If in parts (b), (c), and (d), you were to use Stirling's approximation,

$$K! \sim K^K e^{-K} \sqrt{2\pi K} \,,$$

how would your answers have differed?

Solution :

(a) Since each new deck arrives in the same order, we have $p_1 = 1$ while $p_{2,\ldots,52!} = 0$. Therefore $S = 0$.

(b) For completely randomized decks, $p_n = 1/\Omega$ with $n \in \{1, \ldots, \Omega\}$ and $\Omega = 52!$, the total number of possible configurations. Thus, $S_{\text{random}} = \log_2 52! = 225.581$.

(c) After one riffle shuffle, there are $\Omega = \binom{52}{26}$ possible configurations. If all such configurations were equally likely, we would have $(\Delta S)_{\text{riffle}} = \log_2 \binom{52}{26} = 48.817$. However, they are not all equally likely. For example, the probability that we drop the entire left-half deck and then the entire right half-deck is 2^{-26}. After the last card from the left half-deck is dropped, we have no more choices to make. On the other hand, the probability for the sequence LRLR $\cdots$ is 2^{-51}, because it is only after the 51^{st} card is dropped that we have no more choices. We can derive an exact expression for the entropy of the riffle shuffle in the following manner. Consider a deck of $N = 2K$ cards. The probability that we run out of choices after K cards is the probability of the first K cards dropped being all from one particular half-deck, which is $2 \cdot 2^{-K}$. Now let's ask what is the probability that we run out of choices after $(K+1)$ cards are dropped. If all the remaining $(K-1)$ cards are from the right half-deck, this means that we must have one of the R cards among the first K dropped. Note that this R card cannot be the $(K+1)^{\text{th}}$ card dropped, since then all of the first K cards are L, which we have already considered. Thus, there are $\binom{K}{1} = K$ such configurations, each with a probability 2^{-K-1}. Next, suppose we run out of choices after $(K+2)$ cards are dropped. If the remaining $(K-2)$ cards are R, this means we must have 2 of the R cards among the first $(K+1)$ dropped, which means $\binom{K+1}{2}$ possibilities. Note that the $(K+2)^{\text{th}}$ card must be L, since if it were R this would mean that the last $(K-1)$ cards are R, which we have already considered. Continuing in this manner, we conclude

$$\Omega_K = 2\sum_{n=0}^{K} \binom{K+n-1}{n} = \binom{2K}{K}$$

and

$$S_K = -\sum_{a=1}^{\Omega_K} p_a \log_2 p_a = \sum_{n=0}^{K-1} \binom{K+n-1}{n} \cdot 2^{-(K+n)} \cdot (K+n) \,.$$

The results are tabulated below in Table 1. For a deck of 52 cards, the actual entropy per riffle shuffle is $S_{26} = 46.274$.

(d) Ignoring redundancies, we require $k = S_{\text{random}}/(\Delta S)_{\text{riffle}} = 4.62$ shuffles if we assume all riffle outcomes are equally likely, and 4.88 if we use the exact result for the riffle entropy. Since there are no fractional shuffles, we round up to $k = 5$ in both cases. In fact, computer experiments show that the answer is $k = 9$. The reason we are so far off is that we have ignored redundancies, *i.e.* we have assumed that all the states produced by two consecutive riffle shuffles are distinct. They are not! For decks with asymptotically large

K	Ω_K	S_K	$\log_2\binom{2K}{K}$
2	6	2.500	2.585
12	2704156	20.132	20.367
26	4.96×10^{14}	46.274	48.817
100	9.05×10^{58}	188.730	195.851

Table 1: Riffle shuffle results.

numbers of cards $N \gg 1$, the number of riffle shuffles required is $k \simeq \frac{3}{2}\log_2 N$. See D. Bayer and P. Diaconis, *Annals of Applied Probability* **2**, 294 (1992).

(e) Using the first four terms of Stirling's approximation of $\ln K$, *i.e.* out to $\mathcal{O}(K^0)$, we find $\log_2 52! \approx 225.579$ and $\log_2\binom{52}{26} \approx 48.824$.

(2) In problem #1, we ran across Stirling's approximation,

$$\ln K! \sim K\ln K - K + \tfrac{1}{2}\ln(2\pi K) + \mathcal{O}\big(K^{-1}\big) ,$$

for large K. In this exercise, you will derive this expansion.

(a) Start by writing

$$K! = \int\limits_0^\infty dx\, x^K\, e^{-x} ,$$

and define $x \equiv K(t+1)$ so that $K! = K^{K+1}\, e^{-K}\, F(K)$, where

$$F(K) = \int\limits_{-1}^\infty dt\, e^{K f(t)} .$$

Find the function $f(t)$.

(b) Expand $f(t) = \sum_{n=0}^\infty f_n\, t^n$ in a Taylor series and find a general formula for the expansion coefficients f_n. In particular, show that $f_0 = f_1 = 0$ and that $f_2 = -\frac{1}{2}$.

(c) If one ignores all the terms but the lowest order (quadratic) in the expansion of $f(t)$, show that

$$\int\limits_{-1}^\infty dt\, e^{-Kt^2/2} = \sqrt{\frac{2\pi}{K}} - R(K) ,$$

and show that the remainder $R(K) > 0$ is bounded from above by a function which decreases faster than any polynomial in $1/K$.

(d) *For the brave only!* – Find the $\mathcal{O}\big(K^{-1}\big)$ term in the expansion for $\ln K!$.

Solution :

(a) Setting $x = K(t+1)$, we have

$$K! = K^{K+1}\, e^{-K} \int\limits_{-1}^{\infty} dt\, (t+1)^K\, e^{-t} \ ,$$

hence $f(t) = \ln(t+1) - t$.

(b) The Taylor expansion of $f(t)$ is

$$f(t) = -\tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + \ldots \ .$$

(c) Retaining only the leading term in the Taylor expansion of $f(t)$, we have

$$\begin{aligned} F(K) &\simeq \int\limits_{-1}^{\infty} dt\, e^{-Kt^2/2} \\ &= \sqrt{\frac{2\pi}{K}} - \int\limits_{1}^{\infty} dt\, e^{-Kt^2/2} \ . \end{aligned}$$

Writing $t \equiv s+1$, the remainder is found to be

$$R(K) = e^{-K/2} \int\limits_{0}^{\infty} ds\, e^{-Ks^2/2}\, e^{-Ks} < \sqrt{\frac{\pi}{2K}}\, e^{-K/2} \ ,$$

which decreases exponentially with K, faster than any power.

(d) We have

$$\begin{aligned} F(K) &= \int\limits_{-1}^{\infty} dt\, e^{-\frac{1}{2}Kt^2}\, e^{\frac{1}{3}Kt^3 - \frac{1}{4}Kt^4 + \ldots} \\ &= \int\limits_{-1}^{\infty} dt\, e^{-\frac{1}{2}Kt^2} \Big\{ 1 + \tfrac{1}{3}Kt^3 - \tfrac{1}{4}Kt^4 + \tfrac{1}{18}K^2 t^6 + \ldots \Big\} \\ &= \sqrt{\frac{2\pi}{K}} \cdot \Big\{ 1 - \tfrac{3}{4}K^{-1} + \tfrac{5}{6}K^{-1} + \mathcal{O}\big(K^{-2}\big) \Big\} \end{aligned}$$

Thus,

$$\ln K! = K \ln K - K + \tfrac{1}{2}\ln K + \tfrac{1}{2}\ln(2\pi) + \tfrac{1}{12}K^{-1} + \mathcal{O}\big(K^{-2}\big) \ .$$

(3) A six-sided die is loaded so that the probability to throw a three is twice that of throwing a two, and the probability of throwing a four is twice that of throwing a five.

(a) Find the distribution $\{p_n\}$ consistent with maximum entropy, given these constraints.

(b) Assuming the maximum entropy distribution, given two such identical dice, what is the probability to roll a total of seven if both are thrown simultaneously?

Solution :

(a) We have the following constraints:

$$\begin{aligned} X^0(\boldsymbol{p}) &= p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 1 = 0 \\ X^1(\boldsymbol{p}) &= p_3 - 2p_2 = 0 \\ X^2(\boldsymbol{p}) &= p_4 - 2p_5 = 0 \, . \end{aligned}$$

We define

$$S^*(\boldsymbol{p}, \boldsymbol{\lambda}) \equiv -\sum_n p_n \ln p_n - \sum_{a=0}^{2} \lambda_a \, X^{(a)}(\boldsymbol{p}) \, ,$$

and freely extremize over the probabilities $\{p_1, \ldots, p_6\}$ and the undetermined Lagrange multipliers $\{\lambda_0, \lambda_1, \lambda_2\}$. We obtain

$$\frac{\partial S^*}{\partial p_1} = -1 - \ln p_1 - \lambda_0 \qquad\qquad \frac{\partial S^*}{\partial p_4} = -1 - \ln p_4 - \lambda_0 - \lambda_2$$

$$\frac{\partial S^*}{\partial p_2} = -1 - \ln p_2 - \lambda_0 + 2\lambda_1 \qquad\qquad \frac{\partial S^*}{\partial p_5} = -1 - \ln p_5 - \lambda_0 + 2\lambda_2$$

$$\frac{\partial S^*}{\partial p_3} = -1 - \ln p_3 - \lambda_0 - \lambda_1 \qquad\qquad \frac{\partial S^*}{\partial p_6} = -1 - \ln p_6 - \lambda_0 \, .$$

Extremizing with respect to the undetermined multipliers generates the three constraint equations. We therefore have

$$p_1 = e^{-\lambda_0 - 1} \qquad\qquad p_4 = e^{-\lambda_0 - 1} \, e^{-\lambda_2}$$

$$p_2 = e^{-\lambda_0 - 1} \, e^{2\lambda_1} \qquad\qquad p_5 = e^{-\lambda_0 - 1} \, e^{2\lambda_2}$$

$$p_3 = e^{-\lambda_0 - 1} \, e^{-\lambda_1} \qquad\qquad p_6 = e^{-\lambda_0 - 1} \, .$$

We solve for $\{\lambda_0, \lambda_1, \lambda_2\}$ by imposing the three constraints. Let $x \equiv p_1 = p_6 = e^{-\lambda_0 - 1}$. Then $p_2 = x \, e^{2\lambda_1}$, $p_3 = x \, e^{-\lambda_1}$, $p_4 = x \, e^{-\lambda_2}$, and $p_5 = x \, e^{2\lambda_2}$. We then have

$$\begin{aligned} p_3 = 2p_2 \quad &\Rightarrow \quad e^{-3\lambda_1} = 2 \\ p_4 = 2p_5 \quad &\Rightarrow \quad e^{-3\lambda_2} = 2 \, . \end{aligned}$$

We may now solve for x:

$$\sum_{n=1}^{6} p_n = \left(2 + 2^{1/3} + 2^{4/3}\right) x = 1 \quad \Rightarrow \quad x = \frac{1}{2 + 3 \cdot 2^{1/3}} \,.$$

We now have all the probabilities:

$$p_1 = x = 0.1730 \qquad\qquad p_4 = 2^{1/3} x = 0.2180$$
$$p_2 = 2^{-2/3} x = 0.1090 \qquad\qquad p_5 = 2^{-2/3} x = 0.1090$$
$$p_3 = 2^{1/3} x = 0.2180 \qquad\qquad p_6 = x = 0.1730 \,.$$

(b) The probability to roll a seven with two of these dice is

$$\begin{aligned} P(7) &= 2\, p_1\, p_6 + 2\, p_2\, p_5 + 2\, p_3\, p_4 \\ &= 2\left(1 + 2^{-4/3} + 2^{2/3}\right) x^2 = 0.1787 \,. \end{aligned}$$

(4) The probability density for a random variable x is given by the Lorentzian,

$$P(x) = \frac{\gamma}{\pi} \cdot \frac{1}{x^2 + \gamma^2} \,.$$

Consider the sum $X_N = \sum_{i=1}^{N} x_i$, where each x_i is independently distributed according to $P(x_i)$. Find the probability $\Pi_N(Y)$ that $|X_N| < Y$, where $Y > 0$ is arbitrary.

Solution :

As discussed in the Lecture Notes §1.4.2, the distribution of a sum of identically distributed random variables, $X = \sum_{i=1}^{N} x_i$, is given by

$$P_N(X) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\hat{P}(k)\right]^N e^{ikX} \,,$$

where $\hat{P}(k)$ is the Fourier transform of the probability distribution $P(x_i)$ for each of the x_i. The Fourier transform of a Lorentzian is an exponential:

$$\int\limits_{-\infty}^{\infty} dx\, P(x)\, e^{-ikx} = e^{-\gamma|k|} \,.$$

Thus,

$$\begin{aligned} P_N(X) &= \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-N\gamma|k|}\, e^{ikX} \\ &= \frac{N\gamma}{\pi} \cdot \frac{1}{X^2 + N^2\gamma^2} \,. \end{aligned}$$

The probability for X to lie in the interval $X \in [-Y, Y]$, where $Y > 0$, is

$$\mathit{\Pi}_N(Y) = \int\limits_{-Y}^{Y} dX \, P_N(X) = \frac{2}{\pi} \, \tan^{-1}\!\left(\frac{Y}{N\gamma}\right) .$$

The integral is easily performed with the substitution $X = N\gamma \tan\theta$. Note that $\mathit{\Pi}_N(0) = 0$ and $\mathit{\Pi}_N(\infty) = 1$.