PHYSICS 140A : STATISTICAL PHYSICS
FINAL EXAMINATION

100 POINTS TOTAL

(1) Consider a system of N independent, distinguishable $S = 1$ objects, each described by the Hamiltonian

$$\hat{h} = \Delta\, S^2 - \mu_0 \mathsf{H}\, S \ ,$$

where $S \in \{-1, 0, 1\}$.

(a) Find $F(T, \mathsf{H}, N)$.
[10 points]

(b) Find the magnetization $M(T, \mathsf{H}, N)$. .
[5 points]

(c) Find the zero field susceptibility, $\chi(T) = \frac{1}{N} \frac{\partial M}{\partial \mathsf{H}}\Big|_{\mathsf{H}=0}$.
[5 points]

(d) Find the zero field entropy $S(T, \mathsf{H} = 0, N)$. (*Hint : Take* $H \to 0$ *first.*)
[5 points]

(2) A classical gas consists of particles of two species: A and B. The dispersions for these species are

$$\varepsilon_{\rm A}(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} \qquad , \qquad \varepsilon_{\rm B}(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{4m} - \Delta \ .$$

In other words, $m_{\rm A} = m$ and $m_{\rm B} = 2m$, and there is an additional energy offset $-\Delta$ associated with the B species.

(a) Find the grand potential $\Omega(T, V, \mu_{\rm A}, \mu_{\rm B})$.
[10 points]

(b) Find the number densities $n_{\rm A}(T, \mu_{\rm A}, \mu_{\rm B})$ and $n_{\rm B}(T, \mu_{\rm A}, \mu_{\rm B})$.
[5 points]

(c) If $2{\rm A} \rightleftharpoons {\rm B}$ is an allowed reaction, what is the relation between $n_{\rm A}$ and $n_{\rm B}$?
(*Hint : What is the relation between* $\mu_{\rm A}$ *and* $\mu_{\rm B}$*?*)
[5 points]

(d) Suppose initially that $n_{\rm A} = n$ and $n_{\rm B} = 0$. Find $n_{\rm A}$ in equilibrium, as a function of T and n and constants.
[5 points]

(3) A branch of excitations for a three-dimensional system has a dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{2/3}$. The excitations are bosonic and are not conserved; they therefore obey photon statistics.

(a) Find the single excitation density of states per unit volume, $g(\varepsilon)$. You may assume that there is no internal degeneracy for this excitation branch.
[10 points]

(b) Find the heat capacity $C_V(T,V)$.
[5 points]

(c) Find the ratio E/pV.
[5 points]

(d) If the particles are bosons with number conservation, find the critical temperature $T_{\rm c}$ for Bose-Einstein condensation.
[5 points]

(4) Short answers:

(a) What are the conditions for a dynamical system to exhibit Poincaré recurrence?
[3 points]

(b) Describe what the term *ergodic* means in the context of a dynamical system.
[3 points]

(c) What is the microcanonical ensemble? [3 points]

(d) A system with $L = 6$ single particle levels contains $N = 3$ bosons. How many distinct many-body states are there? [3 points]

(e) A system with $L = 6$ single particle levels contains $N = 3$ fermions. How many distinct many-body states are there? [3 points]

(f) Explain how the Maxwell-Boltzmann limit results, starting from the expression for $\Omega_{\rm BE/FD}(T,V,\mu)$. [3 points]

(g) For the Dieterici equation of state, $p\,(1-bn) = nk_{\rm B}T\exp(-an/k_{\rm B}T)$, find the second virial coefficient $B_2(T)$. [3 points]

(h) Explain the difference between the Einstein and Debye models for the specific heat of a solid. [4 points]

(i) Who composed the song *yerushalayim shel zahav*? [50 quatloos extra credit]