# Solutions to Midterm, UCSD Physics 130b

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November 6, 2012

1a) A spin-$\frac{1}{2}$ particle, such as an electron, proton or neutron, has two possible azimuthal components of spin, $m_s$, $+\frac{1}{2}$ and $-\frac{1}{2}$. Thus, the spinor must have two components to represent these possible states, sometimes referred to as up and down.

1b) A spin-1 particle, such as a photon, has three possible azimuthal components of spin, $m_s$; -1,0,+1. Thus, the spinor must have three components to represent these possible states.

1c), see problem 4.30 in Griffiths. To find the spin component along a particular axis, first construct a spin operator whose eigenvalues are $\pm\frac{\hbar}{2}$ by projecting $\vec{S} = (S_x, S_y, S_z)$ along the axis of interest, $\hat{n}$. Here $\hat{n} = (\sin\theta, 0, \cos\theta) = \frac{1}{\sqrt{2}}(1, 0, 1)$. The projection is given by,

$$S_n = \boldsymbol{S} \cdot \hat{n} = \frac{\hbar}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

We are interested in the eigenstate of $-\frac{\hbar}{2}$. Thus, the corresponding (normalized) eigenvector is

$$\chi_-^{(\hat{n})} = \frac{1}{\sqrt{2}} \begin{pmatrix} +\sqrt{(}1 - \frac{1}{\sqrt{2}}) \\ -\sqrt{(}1 + \frac{1}{\sqrt{2}}) \end{pmatrix} \tag{2}$$

To get the probability that we measured it to have a value $+\frac{\hbar}{2}$ along the x-direction, i.e. that it is in the $\chi_+^{(\hat{x})} = \frac{1}{\sqrt{2}}\binom{1}{1}$ state,

$$P = |\chi_+^{*(\hat{x})} \cdot \chi_-^{(\hat{n})}|^2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \approx 0.15 \tag{3}$$

1d) The Hamiltonian of a particle with spin at rest is given by $H = -\vec{\mu} \cdot \vec{B}$. Thus for spin-$\frac{1}{2}$, with magnetic field $\vec{B} = B_0(0, \sin\alpha, \cos\alpha)$

$$H = -\gamma\vec{S} \cdot \vec{B} = \gamma B_0 \frac{\hbar}{2} \begin{pmatrix} -\cos\alpha & i\sin\alpha \\ -i\sin\alpha & \cos\alpha \end{pmatrix} \tag{4}$$

1e) Similar to problem 1d, we construct the Hamiltonian for a magnetic field along the y-direction.

$$H = -\gamma\vec{S} \cdot \vec{B} = -\gamma B_0 S_y = -\gamma B_0 \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{5}$$

Notice that the Hamiltonian here is proportional to the y-component of the spin operator. Thus, it will have the same eigenstates and eigenvalues scaled by the multiplicative constant;

$$E_{\pm} = \mp\gamma B_0 \frac{\hbar}{2} \tag{6}$$

$$\psi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{7}$$

2a) The fine structure correction is the result of two perturbations of equal magnitude. First, the relativistic correction

$$H'_r = -\frac{p^4}{8m^3c^3} \rightarrow E^{(1)}_{r(n,l)} = -\frac{(E^{(0)}_n)^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] \tag{8}$$

Second, the spin-orbit correction, due to the interaction of the magnetic field (that is seen in the electron's reference frame due to its angular momentum around a radial electric field created by the proton) and the electron's spin,

$$H'_{so} = \frac{e^2}{8\pi\epsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L} \rightarrow E^{(1)}_{so(n,l,j)} = -n\frac{(E^{(0)}_n)^2}{mc^2}\left[\frac{j(j+1)-l(l+1)-\frac{3}{4}}{l(l+\frac{1}{2})(l+1)}\right] \tag{9}$$

2b) If we apply a large magnetic field (in comparison to the Fine Structure perturbation), the magnetic moments (due to the spin and angular momentum of the electron) will align with this magnetic field. Thus, we say the "good" states are those described by quantum numbers $m_l$ and $m_s$. In this case the relativistic contribution remains the same, while the spin-orbit coupling is restricted to contribution from $S_z$ and $L_z$.

2c) Combining the relativistic and spin-orbit energy correction, we see that

$$E^{(1)}_{fs(n,j)} = E^{(0)}_n \left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] \tag{10}$$

From this, we see that we still have a degeneracy of states, namely within j. The degeneracy will be $2j+1$, due to all the possible $m_j$ states. Recall, that for the unperturbed hydrogen, the degeneracy was $2\text{n}^2$, if we took into account the spin degeneracy. Thus, for the ground state of Hydrogen, the degeneracy remains the same and is 2.

2d) To lift or break the degeneracy means to split indistinguishable states, states with the same eigenvalue, into states with different eigenvalues.

2e) We can lift the degeneracy of the Fine Structure correction by applying an external magnetic field. This is known as the Zeeman splitting.

3a) The Hyperfine structure of the hydrogen atom is the result of spin-spin interaction between the electron's spin and the nucleus (proton) spin. Since both spins give a magnetic dipole, this is a magnetic dipole-dipole interaction.

3b) If we consider only the Hyperfine correction, the ground state will split into two states; the Singlet and the Triplet. The degeneracy is clear from their names and is also given by 2f+1. The new ground state is the Singlet state.

3c) Again, this degeneracy can by split by the application of an external magnetic field.

3d) Although, the energy corrections for hyperfine is on the order of $\frac{m_e}{m_p}\alpha^4 mc^2$, which is smaller than the fine structure correction by the factor $\frac{m_e}{m_p} \approx \frac{1}{2000}$, we can treat the Hyperfine correction as a separate effect in the case for states with angular momentum equal to zero. In this case, the spin quantum number describes the good states.

4) See 6.1 of Griffiths.