PHYS 4D
Solution to HW 9

March 6, 2011

Problem Giancoli 36-64 (II) A certain galaxy has a Doppler shift given by $f_0 - f = 0.697 f_0$, How fast is it moving away from us?

Solution:

$$f_0 - f = 0.697 f_0 \Rightarrow \frac{f}{f_0} = 0.303.$$

The galaxy is moving away from the Earth. Eq. 36-15b $\Rightarrow$

$$f = f_0 \sqrt{\frac{c - v}{c + v}} \Rightarrow v = \frac{1 - \frac{f^2}{f_0^2}}{1 + \frac{f^2}{f_0^2}} c = 0.83c.$$

Problem Giancoli 36-66 (II) Starting from Eq. 36-15a, show that the Doppler shift in wavelength is

$$\frac{\Delta \lambda}{\lambda} = \frac{v}{c}$$

if $v \ll c$.

Solution: Eq. 36-15a:

$$\lambda = \lambda_0 \sqrt{\frac{c + v}{c - v}}$$

$$\begin{aligned}
\frac{\lambda - \lambda_0}{\lambda} &= 1 - \sqrt{\frac{c - v}{c + v}} = 1 - \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = 1 - \left(1 - \frac{v}{c}\right)^{\frac{1}{2}} \left(1 + \frac{v}{c}\right)^{-\frac{1}{2}} \\
&\approx 1 - \left(1 - \frac{v}{c}\right) = \frac{v}{c}.
\end{aligned}$$

Problem Giancoli 36-68 (III) A certain atom emits light of frequency f_0 when at rest. A monatomic gas composed of these atoms is at temperature T. Some of the gas atoms move toward and others away from an observer due to their random thermal motion. Using the rms speed of thermal motion, show that the fractional difference between the Doppler-shifted frequencies for atoms moving directly toward the observer and directly away from the observer is $\Delta f / f_0 \approx 2\sqrt{3kT/mc^2}$; assume $mc^2 \gg 3kT$. Evaluate $\Delta f / f_0$ for a gas of hydrogen atoms at $550K$. [This "Doppler-broadening" effect is commonly used to measure gas temperature, such as in astronomy.]

Solution: The difference between the Doppler-shifted frequencies for atoms moving directly toward the observer and directly away from the observer is

$$\begin{aligned}
\Delta f &= f_0 \sqrt{\frac{c + v}{c - v}} - f_0 \sqrt{\frac{c - v}{c + v}} \\
&= f_0 \frac{2v}{\sqrt{(c - v)(c + v)}} = f_0 \frac{2\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}.
\end{aligned}$$

Using the rms speed of thermal motion,

$$v = v_{rms} = \sqrt{\frac{3kT}{m}} \Rightarrow \frac{v}{c} = \sqrt{\frac{3kT}{mc^2}} \Rightarrow$$

$$\Delta f = f_0 \frac{2\sqrt{\frac{3kT}{mc^2}}}{\sqrt{1 - \frac{3kT}{mc^2}}} = f_0 \frac{2}{\sqrt{\frac{mc^2}{3kT} - 1}}$$

assume $mc^2 \gg 3kT$,

$$\Delta f / f_0 \approx 2\sqrt{\frac{3kT}{mc^2}}.$$

Problem Griffiths 12-30 Suppose you have a collection of particles, all moving in the x direction, with energies $E_1, E_2, E_3, \ldots$ and momenta $p_1, p_2, p_3, \ldots$. Find the velocity of the center of momentum frame, in which the total momentum is zero.

Solution:

$$\begin{aligned} E_T &= E_1 + E_2 + E_3 + \ldots; \\ p_T &= p_1 + p_2 + p_3 + \ldots; \\ \bar{p}_T &= \gamma(p_T - \beta E_T/c) = 0 \Rightarrow \beta = v/c = p_T c/E_T. \\ v &= p_T c^2/E_T = c^2(p_1 + p_2 + p_3 + \ldots)/(E_1 + E_2 + E_3 + \ldots). \end{aligned}$$

Problem Griffiths 12-33 A neutral pion of (rest) mass m and momentum $p = (3/4)mc$ decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the energy of each photon.

Solution: Pion's energy:

$$E^2 = p^2c^2 + m^2c^4 = \frac{9}{16}m^2c^4 + m^2c^4 = \frac{25}{16}m^2c^4 \Rightarrow E = \frac{5}{4}mc^2.$$

Conservation of energy:

$$\frac{5}{4}mc^2 = E_A + E_B.$$

Conservation of momentum:

$$\frac{3}{4}mc^2 = p_A + p_B = \frac{E_A}{c} - \frac{E_B}{c}$$

$\Rightarrow$

$$2E_A = 2mc^2.$$

$\Rightarrow$

$$E_A = mc^2, E_B = \frac{1}{4}mc^2.$$

Problem Griffiths 12-35 In a pair annihilation experiment, an electron (mass m) with momentum p_e hits a positron (same mass, but opposite charge) at rest. They annihilate, producing two photons.
(a) Why couldn't they produce just one photon?
(b) If one of the photons emerges at $60°$ to the incident electron direction, what is its energy?

Solution:
(a) One photon is impossible, because in the center of momentum frame, we'd be left with a photon at rest, whereas photons have to travel at speed c.
(b) Conservation of energy:

$$\sqrt{p_e^2 c^2 + m^2 c^4} + mc^2 = E_A + E_B.$$

Conservation of momentum:
horizontal:

$$p_e = \frac{E_A}{c}\cos 60° + \frac{E_B}{c}\cos\theta \Rightarrow E_B\cos\theta = p_e c - \frac{1}{2}E_A,$$

vertical:

$$0 = \frac{E_A}{c} \sin 60° - \frac{E_B}{c} \sin \theta \Rightarrow E_B \sin \theta = \frac{\sqrt{3}}{2} E_A.$$

Square and add $\Rightarrow$

$$
\begin{aligned}
E_B^2 &= \left(p_e c - \frac{1}{2} E_A\right)^2 + \left(\frac{\sqrt{3}}{2} E_A\right)^2 \\
&= p_e^2 c^2 - p_e c E_A + E_A^2 \\
E_B^2 &= \left[\sqrt{p_e^2 c^2 + m^2 c^4} + mc^2 - E_A\right]^2 \\
&= p_e^2 c^2 + m^2 c^4 + 2\sqrt{p_0^2 c^2 + m^2 c^4}\left(mc^2 - E_A\right) + m^2 c^4 - 2mc^2 E_A + E_A^2
\end{aligned}
$$

$\Rightarrow$

$$E_A = mc^2 \frac{mc^2 + \sqrt{p_0^2 c^2 + m^2 c^4}}{mc^2 + \sqrt{p_0^2 c^2 + m^2 c^4} - p_e c/2} = \frac{mc^2}{2} \frac{mc + 2p_e + \sqrt{p_0^2 + m^2 c^2}}{mc + \frac{3}{4} p_0}$$