Electromagnetic waves

Maxwell equations:

$$\nabla \times \vec{B} = \mu \vec{j} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{3}$$

$$\nabla \cdot \vec{B} = 0 \tag{4}$$

By applying derivative $\partial / \partial t$ to Eq. (1) and $\vec{\nabla} \times$ to Eq. (2) we obtain:

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \vec{j}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{5}$$

Vector algebra:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

Similarly

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \tag{6}$$

By using (6) and (3) Eq. (5) can be written in the form:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{j}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho \tag{7}$$

The first term in l.-h. side of (7) contains the vector Laplace operator.

In rectangular (cortesian) coordinates it is given by

$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \vec{i} + \left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \vec{j}$$

$$+ \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \vec{k} = \vec{i} \nabla^2 E_x + \vec{j} \nabla^2 E_y + \vec{k} \nabla^2 E_z$$

Outside of the region with sources Eq. (7) is reduced to ***the wave equation***

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad (8)$$

For 1-d case $\left(\vec{E} = \left[0, E_y(x), 0\right]\right)$ in vacuum (8) is reduced to

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad (9)$$

Let us assume that the solution of Eq. (9) has the form

$$E_y(x,t) = f(x - Ut) \qquad (10)$$

The factor v is a constant. The function f can be any function of a single variable. The purpose of writing $E_y(x,t)$ as we have in (10) is to make the waveform move as a unit in the positive-x direction as time passes. We know that if $f(x)$ is any function of x, then $f(x - x_0)$ is the same function, shifted to the right a distance x_0 along the x axis. If instead of $f(x - x_0)$ we write $f(x - Ut)$, then the function is shifted to the right a distance Ut. This distance increases as time increases, so the function is displaced steadily further out the x axis. The displacement is by a distance Ut, which means that the velocity of motion is U. It is easy to see that the entire waveform travels as a unit with velocity U.

To show that waves can propagate in vacuum, we need to verify that the wave (10) satisfies the wave equation (9). Differentiating (10), we see that $\partial^2 E_y / \partial x^2 = f''(x - Ut)$, where f'' is the second derivative of f with respect to its argument. Similarly $\partial^2 E_y / \partial t^2 = U^2 f''(x - Ut)$. Substituting into (9) we see that the wave equation is satisfied, provided that

$$U^2 = \frac{1}{\varepsilon_0 \mu_0} \qquad (11)$$

A sinusoidal solution of equation (9) describing a traveling wave moving in the positive-x direction can be written as

$$E_y(x,t) = E_0 \cos[k(x - Ut)] \qquad (12)$$

The constant k is the wave number. We see that at a fixed position, E_y varies sinusoidally in time with an angular frequency

$$\omega = kU. \qquad (13)$$

In terms of k and ω

$$E_y(x,t) = E_0 \cos(kx - \omega t) \tag{14}$$

By differentiating any phase of sinusoid in (14) with respect to time, it is easy to see that the velocity U is the velocity of motion of constant phase. *We will use below the symbol* v_{ph} *instead of* U *for the* ***wave phase velocity***.

Thus, solution (14) describes an electromagnetic wave propagating with *phase velocity*

$v_{ph} = \omega/k = 1/\sqrt{\varepsilon_0 \mu_0} \equiv c$ - the speed of light.

Similarly solution of (9) in the form

$$E_y(x,t) = f_1(x + Ut) \tag{10'}$$

and

$$E_y(x,t) = E_0 \cos(kx + \omega t) \tag{14'}$$

describe the wave traveling in the negative-x direction.

The wave equation in a medium that is characterized by ε and μ in 1-d case has a form

$$\frac{\partial^2 E_y}{\partial t^2} - \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} = 0 \tag{15}$$

Sinusoidal solution in this case describes an electromagnetic wave propagating with phase velocity

$$v_{ph} = \omega/k = 1/\sqrt{\varepsilon\mu} = c/\sqrt{K_E K_M} < c \tag{16}$$

Now consider the wave propagation in 3-d case that is described by Eq. (8).

In our study of sinusoidal EM waves in this case we will use a more general approach by using the ***phasor*** techniques.

PHASORS

Phasor is a complex number that represents a sinusoidal function of time.

Let us consider a sinusoidal function of time

$$B(t) = A_0 \cos(\varphi + \omega t)$$

It can be represented in the following form

$$B(t) = A_0 \cos(\varphi + \omega t) = \mathrm{Re}\left\{A_0 e^{i\varphi} e^{i\omega t}\right\} = \mathrm{Re}\left\{\underline{B} e^{i\omega t}\right\}$$

The complex quantity $\underline{B} = A_0 e^{i\varphi}$ is the phasor of the sinusoidal function B(t).

Phasors contain information about amplitude and phase of the sinusoids.

RULE 1: *If a sinusoid is described by formula* $E = A\cos(kx + \omega t)$ *the phasor representing the sinusoid is* $\underline{E} = Ae^{ikx}$

Example: $B(t) = A\sin(kx + \omega t)$. Find $\underline{B}$.

$$B(t) = A\sin(kx + \omega t) = A\cos(kx + \omega t - \pi/2) \Rightarrow \underline{B} = e^{ikx - i\pi/2}$$

RULE 2: *To obtain the sinusoid corresponding to a given phasor, multiply the phasor by* $e^{i\omega t}$ *and take the real part. Thus the sinusoid corresponding to the phasor* $\underline{E}$ *is* $\underline{\mathrm{Re}\left\{\underline{E} e^{i\omega t}\right\}}$.

Example: $\underline{E} = 5e^{i30^\circ}$. Find E(t).

$$E(t) = \underline{\mathrm{Re}\left\{\underline{E} e^{i\omega t}\right\} = 5\cos(30^\circ} + \omega t)$$

RULE 3: *If* $\underline{E}$ ***is*** *the phasor of the sinusoid* $E(t)$**,** *then the phasor representing the sinusoid* $\partial E(t)/\partial t$ *is* $i\omega \underline{E}$.

To prove rule 3 let us consider a sinusoidal function $B(t) = A_0 \cos(\varphi + \omega t)$ that is represented by a phasor $\underline{B} = A_0 e^{i\varphi}$. Let us find the phasor of the $\partial B(t)/\partial t$.

$$\partial B(t)/\partial t = -\omega A_0 \sin(\varphi + \omega t) = -\omega A_0 \cos(\varphi + \omega t - \pi/2)$$

According to Rule 1, the phasor of this sinusoidal function can be written as

$$\underline{\partial B(t)/\partial t} = -\omega A_0 \exp[i(\varphi - \pi/2)] = i\omega A_0 e^{i\varphi}$$

Phasor analysis is used for study of sinusoidal signals in linear approximation when all terms in equations have the same frequency.

It is possible to express any wave as a superposition of harmonics with different frequencies.

Then for each harmonic the wave equation

$$\nabla^2 \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{17}$$

Can be written in the form of equation for the phasor

$$\nabla^2 \vec{E} + \omega^2 \varepsilon\mu \vec{E} = 0 \quad \text{- Helmholtz equation} \tag{18}$$

We omitted here the underline symbol in the phasor $\underline{\vec{E}}$ that represents the sinusoidal function of time $\vec{E}(\vec{r},t)$,

Solution for every component of the phasor $\vec{E}$ can be obtained by using the procedure of separation of variables.

In rectangular (Cartesian) coordinates

$$E_i(x,y,z) = X_i(x) Y_i(y) Z_i(z) \qquad (i = x,y,z) \tag{19}$$

and from Helmholtz equation (18) we obtain

$$\frac{\partial^2 X_i}{\partial x^2}\frac{1}{X_i(x)} + \frac{\partial^2 Y_i}{\partial y^2}\frac{1}{Y_i(y)} + \frac{\partial^2 Z_i}{\partial z^2}\frac{1}{Z_i(z)} + \omega^2 \varepsilon\mu = 0 \tag{20}$$

It follows from Eq. (20) that each of the first three terms must be constant.

$$\frac{\partial^2 X_i}{\partial x^2}\frac{1}{X_i(x)} = -k_x^2; \quad (k_x^2 = const > 0)$$

$$\frac{\partial^2 Y_i}{\partial y^2}\frac{1}{Y_i(y)} = -k_y^2; \quad (k_y^2 = const > 0) \tag{21}$$

$$\frac{\partial^2 Z_i}{\partial z^2}\frac{1}{Z_i(z)} = -k_z^2; \quad (k_z^2 = const > 0)$$

where according to (20) k_x^2, k_y^2, k_z^2 satisfy the so-called ***dispersion equation***

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \varepsilon\mu \tag{22}$$

Solution of the first equation in (21) can be written in the form

$$X_i(x) = A_i e^{-ikx} + B_i e^{ikx} \tag{23}$$

If only first term is retained in phasor (), the sinusoidal in time domain solution for the field will have a form

$$X_i(x,t) = \mathrm{Re}\left\{A_i e^{-i(kx-\omega t)}\right\} = A_i \cos(kx - \omega t) \tag{24}$$

It corresponds to the wave traveling in positive x-direction.

Second term in (23) corresponds to the wave propagating in –x direction. Such a component appears usually due to reflection from some boundary. As a result, (23) describes a standing wave solution that is formed in this case. For example in the simplest case when $A_i = B_i$, the phasor (23) is equal

$$X_i(x) = 2A_i \cos k_x x \tag{25}$$

and the sinusoidal in time domain solution of the first of Equations (21) is just a standing wave

$$X_i(x,t) = 2A_i \cos(k_x x)\cos(\omega t) \tag{26}$$

Solution for a traveling wave can be written in the form

$$\begin{aligned} E_i(t,x,y,z) &= \mathrm{Re}\left\{X_i(x)Y_i(y)Z_i(z)e^{i\omega t}\right\} = E_i \,\mathrm{Re}\left\{-i(k_x x + k_y y + k_z z - \omega t)\right\} \\ &= E_i \cos(\vec{k}\vec{r} - \omega t) \end{aligned} \tag{27}$$

Here $\vec{k} = \langle k_x, k_y, k_z \rangle$ is the wave vector, its direction defines the direction of the wave propagation, plane perpendicular to $\vec{k}$ is the plane of constant phase.

It follows from the dispersion relation (22) that the phase velocity of EM wave in a medium is $v_{ph} = \omega/k = c/\sqrt{\varepsilon\mu}$

that is smaller than light speed in vacuum.