PHYSICS 2D FINAL EXAM WINTER QUARTER 2011
PROF. HIRSCH MARCH 14th, 2011

Formulas:

Time dilation; Length contraction: $\Delta t = \gamma \Delta t' \equiv \gamma\, \Delta t_p$; $L = L_p/\gamma$; $c = 3\times 10^8 m/s$

Lorentz transformation: $x' = \gamma(x - vt)$; $y' = y$; $z' = z$; $t' = \gamma(t - vx/c^2)$; inverse: $v \to -v$

Spacetime interval: $(\Delta s)^2 = (c\Delta t)^2 - [\Delta x^2 + \Delta y^2 + \Delta z^2]$; $\gamma = 1/\sqrt{1 - v^2/c^2}$

Velocity transformation: $u_x' = \dfrac{u_x - v}{1 - u_x v/c^2}$; $u_y' = \dfrac{u_y}{\gamma(1 - u_x v/c^2)}$; inverse: $v \to -v$

Relativistic Doppler shift: $f_{obs} = f_{source}\sqrt{1 + v/c}\ /\ \sqrt{1 - v/c}$ (approaching)

Momentum: $\vec{p} = \gamma m\vec{u}$; Energy: $E = \gamma mc^2$; Kinetic energy: $K = (\gamma - 1)mc^2$

Rest energy: $E_0 = mc^2$; $E = \sqrt{p^2c^2 + m^2c^4}$

Electron: $m_e = 0.511\, MeV/c^2$ Proton: $m_p = 938.26\, MeV/c^2$ Neutron: $m_n = 939.55\, MeV/c^2$

Atomic mass unit: $1\, u = 931.5\, MeV/c^2$; electron volt: $1\text{eV} = 1.6\times 10^{-19}$ J

Stefan's law: $e_{tot} = \sigma T^4$, e_{tot} = power/unit area ; $\sigma = 5.67\times 10^{-8} W/m^2K^4$

$e_{tot} = cU/4$, U = energy density = $\int_0^\infty u(\lambda, T)d\lambda$; Wien's law: $\lambda_m T = \dfrac{hc}{4.96k_B}$

Boltzmann distribution: $P(E) = Ce^{-E/(k_BT)}$

Planck's law: $u_\lambda(\lambda, T) = N_\lambda(\lambda)\times \overline{E}(\lambda, T) = \dfrac{8\pi}{\lambda^4}\times \dfrac{hc/\lambda}{e^{hc/\lambda k_BT} - 1}$; $N(f) = \dfrac{8\pi f^2}{c^3}$

Photons: $E = hf = pc$; $f = c/\lambda$; $hc = 12{,}400\ eV\,A$; $k_B = (1/11{,}600)eV\ /K$

Photoelectric effect: $eV_s = K_{max} = hf - \phi$, $\phi \equiv$ work function; Bragg equation: $n\lambda = 2d\sin\vartheta$

Compton scattering: $\lambda' - \lambda = \dfrac{h}{m_ec}(1 - \cos\theta)$; $\dfrac{h}{m_ec} = 0.0243A$; Coulomb constant: $ke^2 = 14.4\ eV\,A$

Coulomb force: $F = \dfrac{kq_1q_2}{r^2}$; Coulomb potential: $V = \dfrac{kq}{r}$; Coulomb energy: $U = \dfrac{kq_1q_2}{r}$

Force in electric and magnetic fields (Lorentz force): $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

Rutherford scattering: $\Delta n = C\dfrac{Z^2}{K_\alpha^2}\dfrac{1}{\sin^4(\phi/2)}$; $\hbar c = 1{,}973\ eV\,A$

Hydrogen spectrum: $\dfrac{1}{\lambda_{mn}} = R(\dfrac{1}{m^2} - \dfrac{1}{n^2})$; $R = 1.097\times 10^7\ m^{-1} = \dfrac{1}{911.3A}$

Bohr atom: $E_n = -\dfrac{ke^2Z}{2r_n} = -E_0\dfrac{Z^2}{n^2}$; $E_0 = \dfrac{ke^2}{2a_0} = \dfrac{m_e(ke^2)}{2\hbar^2} = 13.6eV$; $K = \dfrac{m_ev^2}{2}$; $U = -\dfrac{ke^2Z}{r}$

$hf = E_i - E_f$; $r_n = r_0n^2$; $r_0 = \dfrac{a_0}{Z}$; $a_0 = \dfrac{\hbar^2}{m_eke^2} = 0.529A$; $L = m_evr = n\hbar$ angular momentum

de Broglie: $\lambda = \dfrac{h}{p}$; $f = \dfrac{E}{h}$; $\omega = 2\pi f$; $k = \dfrac{2\pi}{\lambda}$; $E = \hbar\omega$; $p = \hbar k$; $E = \dfrac{p^2}{2m}$

Wave packets: $y(x,t) = \sum_j a_j\cos(k_jx - \omega_jt)$, or $y(x,t) = \int dk\, a(k)\, e^{i(kx - \omega(k)t)}$; $\Delta k\Delta x \sim 1$; $\Delta\omega\Delta t \sim 1$

group and phase velocity : $v_g = \dfrac{d\omega}{dk}$; $v_p = \dfrac{\omega}{k}$; Heisenberg: $\Delta x\Delta p \sim \hbar$; $\Delta t\Delta E \sim \hbar$

Schrodinger equation: $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\dfrac{\partial\Psi}{\partial t}$; $\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$

Time – independent Schrodinger equation: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi(x) = E\psi(x)$; $\int_{-\infty}^{\infty} dx\ |\psi(x)|^2 = 1$

∞ square well: $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$; $E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}$; $\frac{\hbar^2}{2m_e} = 3.81 eVA^2$ (electron)

Harmonic oscillator: $\Psi_n(x) = H_n(x)e^{-\frac{m\omega}{2\hbar}x^2}$; $E_n = (n+\frac{1}{2})\hbar\omega$; $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$; $\Delta n = \pm 1$

Expectation value of $[Q]$: $<Q> = \int \psi^*(x)[Q]\psi(x)\,dx$; Momentum operator: $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Eigenvalues and eigenfunctions: $[Q]\,\Psi = q\,\Psi$ (q is a constant) ; uncertainty: $\Delta Q = \sqrt{<Q^2> - <Q>^2}$

Step potential: reflection coef: $R = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$, $T = 1 - R$; $k = \sqrt{\frac{2m}{\hbar^2}(E-U)}$

Tunneling: $\psi(x) \sim e^{-\alpha x}$; $T = e^{-2\alpha\Delta x}$; $T = e^{-2\int_{x_1}^{x_2}\alpha(x)dx}$; $\alpha(x) = \sqrt{\frac{2m[U(x)-E]}{\hbar^2}}$

Schrodinger equation in 3D: $-\frac{\hbar^2}{2m}\nabla^2\Psi + U(\vec{r})\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi}{\partial t}$; $\Psi(\vec{r},t) = \psi(\vec{r})e^{-i\frac{E}{\hbar}t}$

3D square well: $\Psi(x,y,z) = \Psi_1(x)\Psi_2(y)\Psi_3(z)$; $E = \frac{\pi^2\hbar^2}{2m}(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2})$

Spherically symmetric potential: $\Psi_{n,\ell,m_\ell}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^{m_\ell}(\theta,\phi)$; $Y_\ell^{m_\ell}(\theta,\phi) = P_\ell^{m_\ell}(\theta)e^{im_\ell\phi}$

Angular momentum: $\vec{L} = \vec{r}\times\vec{p}$; $[L_z] = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$; $[L^2]Y_\ell^{m_\ell} = \ell(\ell+1)\hbar^2 Y_\ell^{m_\ell}$; $[L_z] = m\hbar$

Radial probability density: $P(r) = r^2|R_{n\ell}(r)|^2$; Energy: $E_n = -\frac{ke^2}{2a_0}\frac{Z^2}{n^2}$

Ground state of hydrogen and hydrogen - like ions: $\Psi_{1,0,0} = \frac{1}{\pi^{1/2}}(\frac{Z}{a_0})^{3/2}e^{-Zr/a_0}$

Orbital magnetic moment: $\vec{\mu} = \frac{-e}{2m_e}\vec{L}$; $\mu_z = -\mu_B m_l$; $\mu_B = \frac{e\hbar}{2m_e} = 5.79\times10^{-5} eV/T$

Spin 1/2: $s = \frac{1}{2}$, $|S| = \sqrt{s(s+1)}\hbar$; $S_z = m_s\hbar$; $m_s = \pm 1/2$; $\vec{\mu}_s = \frac{-e}{2m_e}g\vec{S}$

Orbital + spin mag moment: $\vec{\mu} = \frac{-e}{2m}(\vec{L} + g\vec{S})$; Energy in mag. field: $U = -\vec{\mu}\cdot\vec{B}$

Two particles : $\Psi(\vec{r}_1,\vec{r}_2) = +/- \Psi(\vec{r}_2,\vec{r}_1)$; symmetric/antisymmetric

Screening in multielectron atoms: $Z \to Z_{eff}$, $1 < Z_{eff} < Z$

Orbital ordering:

1s < 2s < 2p < 3s < 3p < 4s < 3d < 4p < 5s < 4d < 5p < 6s < 4f < 5d < 6p < 7s < 6d ~ 5f

Justify all your answers to all 6 (six) problems. Write clearly.

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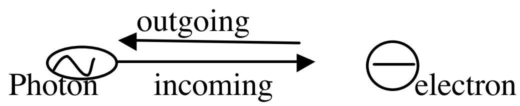
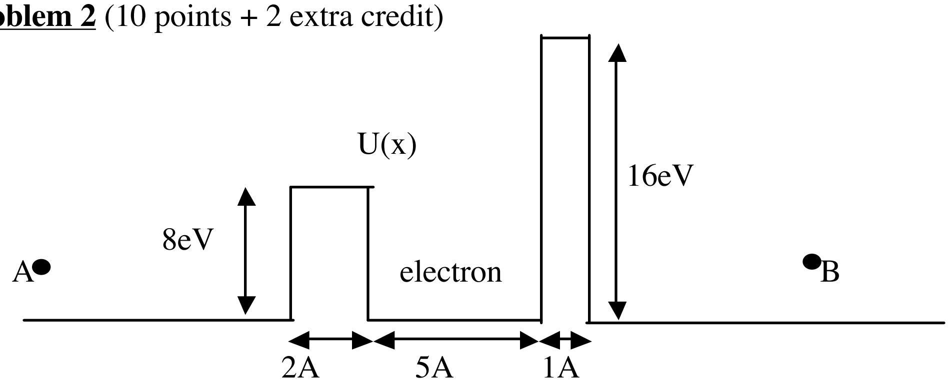
Problem 1 (10 points + 2 extra credit)



A high energy photon hits an electron at rest in free space and bounces back in direction opposite to the incoming direction. The electron acquires a speed v=0.6c.
(a) Find the kinetic energy of the electron using classical mechanics (no relativity) in eV.
(b) Find the kinetic energy of the electron using relativistic relations, in eV.
(c) Which of the answers (a) or (b) is better and why?
(d) Find the momentum of the electron in units eV/c using relativistic relations.
(e) If λ is the wavelength of the incoming photon and λ' is the wavelength of the outgoing photon find the value of $\lambda\lambda'$, in A^2.
(f) For extra credit: find λ and λ', in A.

Problem 2 (10 points + 2 extra credit)



An electron is in the one-dimensional potential well of width 5A shown in the figure. There are potential barriers of width 2A, height 8eV and of width 1A, height 16eV to the left and to the right respectively.
(a) Assume the energy levels for the electrons in this well are approximately the same as the energy levels of an infinite well (box) of the same width. Find the energies of the ground state and of the first excited state, in eV. Use $\hbar^2\pi^2/2m_e = 37.6\ eV\,A^2$ or
$\hbar^2/2m_e = 3.81 eV\,A^2$
(b) Explain why there is no second excited state for the electron in this well.
(c) What is the longest wavelength photon that an electron in the ground state of this well can absorb?
(d) The electron has a non-zero probability of escaping this well. Assume it is initially in the ground state. If it does escape, is it more likely to be found to the left of the well (point A) or to the right of the well (point B)? Find the ratio of the probabilities, P_A/P_B.
(e) Same as (d) if the electron is initially in the first excited state. Assume it will escape rather than make a transition to the ground state.
(f) For extra credit: since this is not an infinite well, the energies are not exactly the same as for the infinite well. Find an approximation to the ground state energy of the electron in this well by using the fact that the wavefunction can penetrate the barriers.

Problem 3 (10 points)
An electron in the ground state of a one-dimensional harmonic oscillator potential $U(x) = \frac{1}{2} m_e \omega^2 x^2$ has expectation value $< x^2 >= 2A^2$.
(a) Find the uncertainty in position Δx in A.
(b) Find the uncertainty in momentum, Δp, in eV/c. Use that $\Delta x \Delta p = \hbar/2$.
(c) Find the average kinetic energy in eV.
(d) Using the fact that the average kinetic energy equals the average potential energy find the value of $\hbar\omega$ for this potential, in eV.
(e) Find the value (in A) of the classical amplitude of oscillation for an electron with energy given by the ground state energy. The classical amplitude is the coordinate corresponding to maximum displacement of the oscillator.

Problem 4 (10 points + 2 extra credit)
The wave function for an electron in a hydrogen-like ion with Z=3 is given by
$\psi(r,\theta,\phi) = Cr^2 e^{-r/a_0} \sin^2\theta \, e^{-2i\phi}$
with C a constant.
(a) Give the values of the quantum numbers n, ℓ, m_ℓ for this electron and its energy in eV. Justify clearly how you know the values for the quantum numbers for this ψ.
(b) Find the wavelengths of all the photons that will be emitted when this electron makes transitions to lower energy states. Include allowed transitions (those where ℓ changes by 1 or -1) and `forbidden' transitions (those that don't satisfy that selection rule, which are possible but less likely).
(c) How much more likely is it to find this electron at a point with $\theta = \pi/2$ than at a point with $\theta = \pi/4$ with the same values for r and ϕ?
(d) Assume a magnetic field of magnitude 5T pointing in the +z direction is turned on, and that the electron stays in this state. Ignoring the spin of the electron (and its associated magnetic moment), is its energy going to increase or decrease? By how much? Give the answer in eV.
(e) <u>For extra credit:</u> repeat (d) taking into account the spin of the electron, and find two different answers depending on the spin orientation.

Problem 5 (10 points)
A two-dimensional box has side lengths L_1=1A, L_2=$(1/\sqrt{2})$A. For an electron in this box, ignoring spin:
(a) Find the quantum numbers of the 5 lowest energy levels and give the ratio of each of the energies to the ground state energy.
(b) Assuming the dimensions L_1 and L_2 correspond to the x and y directions respectively, give the quantum numbers and energies (in eV) of the two lowest energy states where the wave function is zero on the line x=L_1/2. Write down the wavefunctions for these states.
(c) If there are 9 electrons in this box and assuming no interaction between electrons: find the total energy of the system, in eV. Indicate which energy levels are occupied by how many electrons.

Problem 6 (10 points + 2 extra credit)

The radial wave function for an electron in the $n = 2,\ \ell = 1$ state of the hydrogen atom is
$R(r) = Cre^{-r/2a_0}$
with C a constant. The radial probability is $P(r)=r^2R^2(r)$.
(a) Find the most probable r for this electron.
(b) Compare the answer in (a) with the value of the radius of the orbit in the Bohr model for n=2. Explain why they are the same or different.
(c) Find the uncertainty in the radius $\Delta r = \sqrt{<r^2> - <r>^2}$ for the electron described by this wave function. If it is zero, explain why.
(d) Find the value of the momentum $p=m_e v$ for the electron in the n=2 state in the Bohr model. Express your answer in terms of a_0 (Bohr radius) and $\hbar$.
(e) For extra credit: Using the fact that $<p^2>$ for the wavefunction given above has the same value as p^2 in the Bohr atom for the same n, find the uncertainty in the momentum, Δp. Write $\Delta r\ \Delta p$ for the electron in this state in the form $\Delta r \Delta p = constant \times \hbar$ and give the value of that constant (it is dimensionless). Is the result compatible with the uncertainty principle?

Hint 1: $\int_0^\infty dr\ r^k e^{-\lambda r} = \frac{k!}{\lambda^{k+1}}$

Hint 2: The value of C resulting from normalization is $C^2 = \frac{1}{4!a_0^5}$

Justify all your answers to all problems. Write clearly.