Experiment 3, Physics 2BL

Testing a damping model

Preparation

Before this experiment, we recommend you review
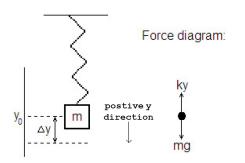or familiarize yourself with the following:
– *Chapters 7 in Taylor*
– *Simple Harmonic Motion*

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1. PHYSICS

For this experiment you will need to be familiar with 3
physical systems and the equations of motion that govern
them. Our method for discussing these systems will be
as follows. First we will set up a force body diagram and
specify initial conditions for the position and velocity.
Then we will determine the equilibrium condition that
occurs when the forces cancel out. When we sum up
the forces and set them equal to mass times acceleration
(Newton's Second Law) we get a differential equation.
After we write the general solution to this differential
equation, we can solve for the constants using the initial
conditions and plot the resulting motion over time.

1.1. Spring Harmonic Oscillator

Force diagram:

The first system is very simple but is a good warm up
problem. We have a mass hanging from a spring as shown
in the figure. The equilibrium position, y_0 is determined
when the forces cancel out:

$$mg = ky_0 \rightarrow y_0 = \frac{mg}{k}$$

We also set the initial conditions so that the initial
position is given a slight displacement, Δy, from equi-
librium. This is similar to the small angle displacement
we used for the simple pendulum, but in this case we are
assuming that the spring constant is not going to change

for a small amount of stretching. Also we are releasing
the mass at rest, so the initial velocity is zero. We will
be using the notation that a dot is a derivative in time.

$$y(t = 0) = y_0 + \Delta y; \qquad \dot{y}(t = 0) = 0$$

We set the forces equal to the mass times acceleration.

$$\Sigma F = -ky + mg = m\ddot{y}$$

And rearrange to form a differential equation.

$$\ddot{y} + \frac{k}{m}y = g$$

Substituting $\omega = \sqrt{\frac{k}{m}}$, we have the following two
solutions. The solutions to the characteristic equation
$r^2 + \frac{k}{m} = 0$ are $r_1 = +i\sqrt{\frac{k}{m}}$ and and $r_2 = -i\sqrt{\frac{k}{m}}$. The
second solution also works if you plug it in, and it takes
care of the equilibrium condition. According to the rules
of differential equations, the total solution is just the sum
of any possible solutions.

$$y_h(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

$$y_p(t) = \frac{mg}{k} = y_0$$

$$y(t) = y_h(t) + y_p(t) = c_1 \cos wt + c_2 \sin wt + y_0$$

Now we substitute our initial conditions to determine
the constants c_1 and c_2.

$$y(0) = c_1 + y_0 = y_0 + \Delta y \qquad \rightarrow c_1 = \Delta y$$

$$\dot{y}(t) = -c_1\omega \sin \omega t + c_2\omega \cos \omega t$$

$$\dot{y}(0) = c_2\omega = 0 \qquad \rightarrow c_2 = 0$$

Plugging in the constants we have the following solu-
tion for the position, which we have plotted below.
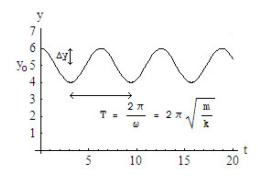
$$y(t) = \Delta y \cos \omega t + y_0$$
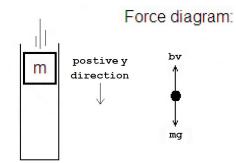
FIG. 1: Motion of a Spring Harmonic Oscillator

Questions

1. What are the differences between this ideal system and a spring oscillator in the presence of real life conditions? Hint: What are we assuming about the spring? Why would we never see motion exactly like the plot in FIG. 1?

2. What are the physical units of the spring constant, k?

3. Say you have two ideal mass on a spring systems. First, suppose both have the same spring constant k, but have different masses, $m_1 < m_2$. How do the periods of these two systems compare? Next, suppose they have different spring constants $k_1 < k_2$, but they both have the same mass m. You raise the mass of each system to the same height and release them. How do the amplitudes compare?

1.2. Mass falling in a drag force

The next system describes a mass in free fall with a drag force from air resistance. In order to vary the drag force we use a tube that has a variable amount of air pressure under the mass. The drag force is a function of the velocity of the falling mass. For mathematical sim-

plicity, we will choose a model in which the force is equal to a constant b times the velocity v. Note that in general damping force may not depend linearly on velocity, but can have quadratic or other functional dependence on it. The purpose of this lab is to test the validity of the assumption that $F_{drag} = -bv$.

Now, there is an equilibrium velocity, analogous to the equilibrium length associated with a spring. We will call this the terminal velocity, v_t, because during free fall the mass will speed up to the terminal velocity and then remain fixed at that speed.

In order to solve for v_t we set the forces equal to each other.

$$bv_t = mg \rightarrow \boxed{v_t = \frac{mg}{b}}$$

We'll set the initial position and velocity at zero because we are dropping the mass from rest from the top of the tube.

$$y(t=0) = 0; \qquad \dot{y}(t=0) = 0$$

We set the forces equal to the mass times acceleration.

$$\Sigma F = -b\dot{y} + mg = m\ddot{y}$$

And rearrange to form a differential equation.

$$\ddot{y} + \frac{b}{m}\dot{y} = g$$

The solutions to the characteristic equation $r^2 + \frac{b}{m}r = 0$ are $r_1 = 0$ and $r_2 = -\frac{b}{m}$. The solutions to the differential equation are thus

$$y_h(t) = c_1 + c_2 e^{-\frac{b}{m}t}$$

$$y_p(t) = \frac{mg}{b}t = v_t t$$

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2 e^{-\frac{b}{m}t} + v_t t$$

Now we substitute our initial conditions to determine the constants c_1 and c_2 and get the following solution for y(t):

$$y(t) = v_t[\frac{m}{b}(e^{-\frac{b}{m}t} - 1) + t]$$

This function is not terribly enlightening because it's just the equation of a mass falling. What is more interesting is if we take a derivative and look at the velocity:

$$\dot{y}(t) = v_t[1 - e^{-\frac{b}{m}t}]$$

The prefactor $\frac{mg}{b}$ determines the final value that the velocity will reach. The constant in the exponent, $\frac{b}{m}$ determines how fast the velocity will reach its final value. This function is plotted in the figure below. These graphs are a plot of the velocity function for four different values of the drag constant, b, while keeping the mass constant.
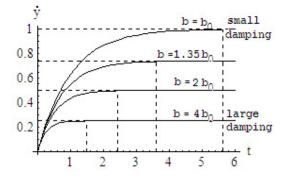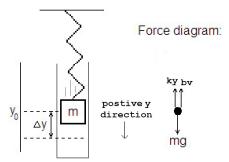

FIG. 2: Velocity of a mass falling in a drag force for varying drag constant.

Questions

4. What are the physical units of the drag constant, b?

5. According to the FIG. 2, what happens to the terminal velocity as the damping is increased? (horizontal lines) What happens to the time it takes for the velocity to reach terminal velocity as the damping is increased? (vertical lines)

1.3. Damped Harmonic Motion

Now we will combine the spring force and the drag force to show what happens when the harmonic motion of a spring is damped. We will use the same initial conditions as the first system. Also the equilibrium position,

$y_0 = \frac{mg}{k}$, will be the same because when the mass is stationary, it feels no drag force.

We set the forces equal to the mass times acceleration.

$$\Sigma F = -ky - b\dot{y} + mg = m\ddot{y}$$

And rearrange to form a differential equation.

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = g$$

The solutions to the characteristic equation $r^2 + \frac{b}{m}r + \frac{k}{m} = 0$ are found using the quadratic formula. This gives us $r = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$ Let's introduce some variables to make this notation look nicer. If we define $\gamma = \frac{b}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ then we get $r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ The damping coefficient b, the mass m, and the spring constant k are parameters of the system that we can control. Since we can give b, m, and k any values we want, in the equation for r, the value under the square root sign can be positive, negative, or zero. These three options give us the three types of solutions to the differential equation. A list of the three cases and the conditions under which they occur is given here.

- Overdamped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} > 0$

- Underdamped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} < 0$

- Critically damped: When $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m} = 0$

Note: In this experiment, m and k are fixed and we only adjust the value of b.
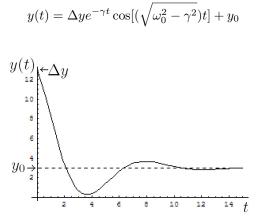
Overdamped

This is the condition that the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is positive. In this case the solution to the differential equation is:

$$y(t) = \frac{\Delta y}{2}e^{-\gamma t}[e^{-(\sqrt{\gamma^2 - \omega_0^2})t} + e^{+(\sqrt{\gamma^2 - \omega_0^2})t}] + y_0$$

How do we interpret this graph physically? This solution/graph tells us, for a system that is overdamped, if we drop the mass from a position Δy above the equilibrium point y_0, then it fall toward the equilibrium position exponentially in time. In other words, it will initially fall toward y_0 quickly, then it will fall more slowly toward y_0 as time passes.

Underdamped

This is the condition that the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is negative. In order to deal with this imaginary solution we can factor out a $\sqrt{-1}$. This switches the values under the radical. As in the simple harmonic case, imaginary solutions to the characteristic equation give oscillatory solutions to the differential equation. In this case the solution to the differential equation is:

$$y(t) = \Delta y e^{-\gamma t} \cos[(\sqrt{\omega_0^2 - \gamma^2})t] + y_0$$

How do we interpret this graph physically? The solution/graph tells us, for a system that is underdamped, if we drop a mass from a position Δy above y_0 then the mass will oscillate about y_0 but the amplitude of this oscillation will decrease exponentially in time. In a non-ideal case the mass will eventually stop at the equilibrium point y_0.
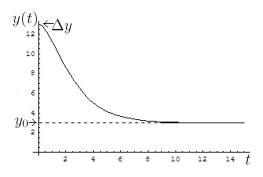
Critically Damped

This is the condition that the value under the radical, $\gamma^2 - \omega_0^2 = \frac{b^2}{4m^2} - \frac{k}{m}$ is zero. Solving for b gives us

$$\boxed{b_{crit} = 2\sqrt{mk}}$$

In this case the solution to the differential equation is simply :

$$y(t) = \Delta y(1 + \gamma t)e^{-\gamma t} + y_0$$

How do we interpret this graph physically? The solution/graph here tell us, for a critically damped system, a

mass dropped from Δy above y_0 will behave much like the overdamped system. It will initially fall quickly toward y_0 and then fall more slowly toward y_0 as time passes. The difference is that in this critically damped system the mass will fall to y_0 in the shortest time possible without oscillating about y_0.

2. METHODS FOR STATISTICAL ANALYSIS

2.1. Plotting graphs

Plotting data makes it easy to visualize trends and in some cases it can end up saving you a lot of work. Here are some guidelines to keep in mind.

1. Give each graph a title.

2. Decide which variable is the independent variable (values you have chosen, i.e. height to drop the mass down the tube) and which is the dependent variable (output of your experiment or a calculated value from that output). Always plot the independent variables on the horizontal axis and the dependent variables on the vertical axis.

3. Figure out what boundaries will be appropriate for each axis, mark off the divisions, and label each axis with its units. Each graph should be about half a page in size.

4. Include vertical error bars for the data points.

2.2. Shortcut method for error propagation

The following method is very useful for saving time in the error propagation process, however, you should only use it when it is appropriate. *It can only be used when the function you are propagating errors is simply a function of variables to given powers.*

$$f(x, y) = Ax^n y^m$$

Starting with the general formula for error propagation we use the rule for taking derivatives of powers, in this case n and m.

$$\sigma_f = \sqrt{(\frac{\partial f}{\partial x}\sigma_x)^2 + (\frac{\partial f}{\partial y}\sigma_y)^2}$$

$$\frac{\partial f}{\partial x} = Anx^{n-1}y^m; \qquad \frac{\partial f}{\partial y} = Amx^ny^{m-1}$$

Now we divide both sides by f. Notice that on the right we have replaced f with its functional form.

$$\frac{\sigma_f}{f} = \sqrt{(\frac{Anx^{n-1}y^m}{Ax^ny^m}\sigma_x)^2 + (\frac{Amx^ny^{m-1}}{Ax^ny^m}\sigma_y)^2}$$

This reduces to a very simple formula.

$$\boxed{\frac{\sigma_f}{f} = \sqrt{(n\frac{\sigma_x}{x})^2 + (m\frac{\sigma_y}{y})^2}} \qquad (1)$$

Notice that this works just as well for cases where n and m are negative numbers. This formula can also be useful because you can compare fractional errors and in some cases rule out errors that are negligible. If you are calculating σ_f, simply multiply both sides by f.

Questions

6. Find an expression for σ_k where $k = \frac{4\pi^2 m}{T^2}$.
First write out the error propagation the long way, using partial derivatives. then divide the left hand side by k and the right hand side by $\frac{4\pi^2 m}{T^2}$ and reduce it to get an expression that only has fractional errors for m and T. You should get the same thing that you would have gotten using the short cut method.

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3. EXPERIMENTAL PROCEDURE

In this experiment you will be testing a model using an "engineering-approach" in which you will need to answer: does the data support the model assumed for damping? The procedure outlined below guides you to compare a computed quantity (model based) with a data-derived quantity, and you will need to derive and reason a conclusion based on this comparison. This procedure assumes damping that is linearly dependent on velocity. After collecting the data, it is up to you to perform a quantitative analysis that tests the validity of this assumption.

The outline of the procedure is:

- First measure the spring constant, k, of a spring.

- From this compute b_{crit}, the damping coefficient required to critically damp a mass m assuming the damping force is linear in velocity.

- Set up an experiment to measure b for a variety of damping levels and adjust it to the critical value.

- Check whether critical damping has been achieved by observing the motion of the spring/mass system within the damping mechanism.

3.1. Spring Harmonic Oscillator

Step 1 You will need a spring, a piston (with eye-hook), and a damping tube. Measure the mass m of your piston. Hang the piston from the spring and measure the period T of small oscillations. In order to increase accuracy, measure the time for N (*at least* 10) oscillations and divide the total time by N . Calculate $k = \frac{4\pi^2 m}{T^2}$ and $b_{crit} = 2\sqrt{mk}$. *Do not overstretch the spring.*
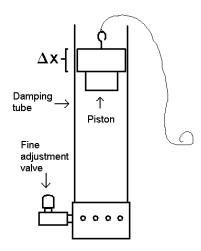
3.2. Damped Free Fall

NOTE: Steps 2 and 3 may take a considerable amount of time. In order to complete the experiment in time, please come well prepared for this experiment and work as quickly and diligently as possible.

Step 2 Measure the thickness of the piston Δx (see diagram below). Set up the photogate at the bottom of the tube and set it to "Gate" mode and 0.1ms precision. This setup will allow you to measure the time Δt it takes for Δx of the piston to fall through the photogate and then to calculate the velocity of the mass at the bottom of the tube ($v = \frac{\Delta x}{\Delta t}$). Orient your piston so the lower half is the cylinder with the smaller diameter. Attach an eye-hook to the top of the piston and tie a string to it so you can pull the piston out of the tube easily. Position the photogate so that it triggers at the beginning and end of Δx of your piston. Be sure it does not trigger from the bottom to the top of the piston nor on the eye-hook. Clamp the tube and photogate down to the table.

Close the valve on the tube. Place a piece of masking tape over 1 hole at the base of the tube so that you have 5 holes OPEN. Choose at least 5 or 6 drop heights. Drop the piston from each height at least 10 times and record the times Δt for each height. Now, place tape over an additional hole so 4 holes remain OPEN and repeat the procedure. Calculate the average times $\bar{\Delta t}$ for each height. *Do not write on the tube.*

Step 3 Convert your time data to velocities using $v = \frac{\Delta x}{\Delta t}$ and make a plot of v vs. h for 5,4 holes *open.* These graphs should look very similar to the graph on page 3 of this lab guide. Because it would be very difficult to measure velocity of something falling as a

function of time, we can measure it as a function of release height. The purpose of this step is to see whether the piston is reaching terminal velocity by the time it is dropped from the top of the tube.

*The following questions, **Q**, are not quiz questions. They are intended to be answered as part of your lab report.*
Q: Will the piston reach v_t for 6 holes? Do we need to check for 0,1,2,3 holes open?

Questions

7. Suppose your damping tube has 3 holes open and you find that your piston does reach terminal velocity when released from a certain height below the maximum release height. Sketch what you would expect a v vs. h plot to look like for this experiment. (h is the release height.) Briefly describe what your plot means.

Step 4   Determine the terminal velocity v_t for 0-5 holes open. This can be done by simply measuring Δt of the piston dropped from the top of the tube for each configuration of holes open. Measure Δt many times and determine the average $\bar{\Delta} t$ for each hole setting.

Step 5   Convert terminal velocity data for 0-5 holes open to b using $b = \frac{mg}{v_t}$ and make a plot of b versus number of holes open. Plot b_{crit} as a horizontal line across your graph.

Q: Where do the two lines intersect? Approximately how many holes need to be open for critical damping?

Step 6   Adjust damping according to your answer to the previous **Q** question. Open the lower number of holes in your range and then use the valve on the tube to open a hole partially. For example, if your estimate was 2.5 holes, open 2 holes and open the valve halfway.

3.3. Damped Harmonic Motion

Step 7   If your b_{crit} is correct, your system should now be close to critically damped. Attach the piston to the spring and observe its motion in the tube. *Make sure the spring is colinear with the tube.*

Q: Does this system look underdamped or overdamped? How can you tell?

Step 8   Next, adjust the valve until the system is critically damped. This is accomplished by visually determining the border between oscillatory and non-oscillatory behavior.

Step 9   Take the piston off the spring, release the piston from the top of the tube, and measure Δt^* N times. Calculate $\bar{\Delta t^*} \pm \sigma_{\bar{\Delta t^*}}$. Δt^* is the time it takes the piston to fall through the photogate at terminal velocity when the tube has been critically damped by eye.

4. ANALYSIS

In order to test the model assumed for damping (linear dependence on velocity), compare the value of spring constant k_{spring} obtained through the more direct method to the value k_{by-eye} obtained by critically damping the system by eye. Find these two values and propagate the errors from m, T, Δt, and Δx.

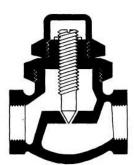$$k_{spring} = \frac{4\pi^2 m}{T^2} \pm \sigma_{k_{spring}}$$

$$k_{by-eye} = m\left(\frac{g\Delta t^*}{2\Delta x}\right)^2 \pm \sigma_{k_{be}}$$

Determine the discrepancy and its uncertainty. Then, using t-score, determine the level at which these two values are discrepant and explain what this means. State your conclusion and explain it!

Questions

8. Three students measure the period of their piston-spring system with the results (in units of seconds):
1.5 ± 0.5
1.17 ± 0.03
1.82 ± 0.19
Determine the best estimate and its uncertainty for the period.

Appendix 1: Lab Equipment, The Damping Tube

In order to damp our spring motion we will be using a long vertical tube that has holes drilled in the bottom. You will be able to cover up the holes in order to increase the pressure underneath the piston that you are damping. The more holes that are open, the less damping, and the piston will be able to move more freely. In order to fine adjust between the levels of holes open, there is a valve that can go anywhere between an open hole and a closed hole. The insides of the valve look something like this. A screw on the side of the tube allows air to flow through.

Cross section of the fine adjustment valve.