2.19 For a spherical celestial object of radius R, the acceleration due to gravity g at a distance x from the center of the object is $g = g_0 R^2/x^2$, where g_0 is the acceleration due to gravity at the object's surface and $x > R$. For the moon, take $g_0 = 1.63$ m/s^2 and $R = 3200$ km. If a rock is released from rest at a height of $5R$ above the lunar surface, with what speed does the rock impact the moon? *Hint: Its acceleration is a function of position and increases as the object falls. So do not use constant acceleration free-fall equations, but go back to basics.*
WARNING: the value given for the initial height may be different for your assignment.
CAUTION: The problem is miss-worded. Instead of "If a rock is released from rest at a height of nR above the lunar surface, with what speed ..." it should say "If a rock is released from rest at a distance of nR from the lunar center, with what speed ..." where n is an integer that however may be different for different students.

ANS: This is one dimensional motion, the motion taking place along a straight line radially outwards from the center of the celestial body. So take the x-axis of coordinates to have origin at the center of the celestial object and be aligned with the motion of the rock we are going to drop. The acceleration is negative since it points towards the origin along the coordinate axis. So the acceleration is $a = -g_0 R^2/x^2$, and we can write the bsic equation of motion:

$$\frac{d^2 x(t)}{dt^2} = -g_0 R^2 \frac{1}{x^2}$$

I hope you got this far on your own. The next step, solving this equation, is not easy. The point is thatthe function $x(t)$ appears on both sides; the $1/x^2$ on the right really is $1/[x(t)]^2$.

Slvingh this requires some trickery. Since we are dropping the rock from some initial distance x_0, it is clear that $x(t)$ will decrease monotonically in time (the rock is falling!). But since $x(t)$ is monotonic we can invert it. That is, there is a function $t = t(x)$ that tells us the time given the instantaneous position of the falling rock. But then I could think of the velocity v and the acceleration a as functions of x. In fact, for the latter we already have an expression that gives it explicitly as a function of x,

This gives us a clue on how to solve the problem. Let's see if we can find v as a function of x (instead of a function of t). Writing the acceleration as the derivative of velocity, and using that the velocity is a function of x, $v = v(x)$, we can use the chain rule for derivatives:

$$\frac{d^2 x(t)}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

This is very nice because any reference to t has disappeared. So we can rewrite the equation of motion as

$$v\frac{dv}{dx} = -g_0 R^2 \frac{1}{x^2}$$

We can integrate this readily. For your table of derivatives you recognize the left hand side as $\frac{d}{dx}\frac{1}{2}v^2$, while the integral of the right hand side involves a power,

$$\int dx\, x^n = \frac{1}{n+1}x^{n+1}$$

Putting these together and including a constant C of integration,

$$\frac{1}{2}v^2 = g_0 R^2 \frac{1}{x} + C.$$

To fix the value of C we use initial conditions: the motion starts from location x_0 with velocity v_0 giving $C = v_0^2/2 - g_0 R^2/x_0$, or

$$\boxed{\frac{1}{2}(v^2 - v_0^2) = g_0 R^2 \left(\frac{1}{x} - \frac{1}{x_0}\right)}$$

If you want to get $x(t)$ you can use here $v = dx/dt$ and integrate. But for our purposes this is enough since we need to compute the velocity as the rock traverses from some height to another.

   You can now do the numerics and plug into your assignment.

   Additional "fun." *OK, I appreciate that this can only be "fun" for a nerd. But you are either an engineering or a chemistry major* ;)   If the initial height of the rock above the lunar surface, $x_0 - R$, is very small compared to the radius of the moon, then we should see the usual constant acceleration result emerge. Indeed,

$$\frac{1}{x} - \frac{1}{x_0} = \frac{x_0 - x}{xx_0} \approx \frac{x - x_0}{R^2}$$

which gives

$$v^2 - v_0^2 = -2g_0(x - x_0)$$

which you must recognize as the correct result for constant acceleration $-g_0$. If you know about Taylor expansions (if you don't, ignore this!), you can find the small corrections to this. For example, if we take $x = R$ and Taylor expand

$$v^2 - v_0^2 = -2g_0(x - x_0) - 4g_0\frac{(x - x_0)^2}{R^2} + \cdots$$