Physics 161: Black Holes: Lecture 9: 24 Jan 2011

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9 Geodesics of Schwarzschild metric from Euler-Langrange

Let's return to our work on geodesics and apply the Euler-Lagrange equations to find the geodesics of the Schwarzschild metric. Remember that the Schwarzschild metric is the unique metric around stationary, spherically symmetric, uncharged objects, so what these geodesics do is tell us how thing move around the Earth, around the Sun, and around uncharged, non-spinning black holes. These are the General Relativistic extension of Newton's and Kepler's laws of motion. Recall the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

As above, we set $c = 1$, and take affine paramter $\lambda = s = \tau$, and extremize $s = \int L d\tau$, with

$$L = 1 = \left[\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{1/2}$$

The Euler-Lagrange equation for t is then

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0,$$

which since $\partial L/\partial t = 0$, implies there is a conserved quantity we will call energy per unit mass:

$$\frac{\partial L}{\partial \dot{t}} \equiv \frac{E}{m}.$$

Calculating,

$$\frac{\partial L}{\partial \dot{t}} = \frac{1}{2}[\cdots]^{-1/2}(1 - \frac{2GM}{r})2\dot{t},$$

where we abbreviated $L = [\cdots]^{1/2}$. Using $L = 1$, we find our t equation

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m}.$$

We don't yet know that the constant has anything to with energy, but we call it E/m, because of our experience with the Minkowski metric. For $r \to \infty$, the Schwarzschild metric goes to the Minkowski metric and for the Minkowski metric $\dot{t} = E/m$.

Next we find the ϕ equation:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0,$$

so again we have a conserved quantity $p_\phi = \partial L/\partial\dot{\phi} \equiv -l/m$, where we will call this conserved quantity the angular momentum per unit mass. Recall Noether's theorem which says if physics is unchanged by a rotation then angular momentum is conserved. Since the metric does not depend explicitly on the angle ϕ, we get that result here. Doing the differentiation we find

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2\sin^2\theta 2\dot{\phi}) = -r^2\sin^\theta\dot{\phi}.$$

Thus our ϕ equation reads

$$\frac{l}{m} = r^2\sin^2\theta\dot{\phi}.$$

Note that it makes sense that we called the constant of motion l/m, since this matches the normal definition of angular momentum, $l = \vec{r}\times\vec{P}$, with $v = r\sin\theta\dot{\phi}$.

Next, we consider the θ equation. Here we find that

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \neq 0,$$

thus we do not have a conserved quantity for this equation. We find

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}[\cdots]^{-1/2}(-r^2\dot{\phi}^2 2\sin\theta\cos\theta) = -r^2\dot{\phi}^2\sin\theta\cos\theta,$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}[\cdots]^{-1/2}(-r^2 2\dot{\theta}) = -r^2\dot{\theta}.$$

Thus our θ equation reads:

$$\frac{d}{d\tau}(r^2\dot{\theta}) = r^2\dot{\phi}^2\sin\theta\cos\theta.$$

Finally we come do the r equation, which is kind of messy because of all the explicit r dependence in the metric. However, we don't have to do it, because we can get the fourth equation we need to specify the equations of motion from our definition of the Lagrangian, $L^2 = 1$:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1-\frac{2GM}{r})} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2.$$

Since our object and metric are spherically symmetric we can simplify things by only considering motion in the equatorial plane ($\theta = \pi/2$, and $\dot{\theta} = 0$). Of course we have to remember that we made this assumption later when we use our equations! If we try to consider motion that has a changing θ, or which is not in this plane we would need to come back to the equation above. In this case then, the equation $L = 1$ becomes:

$$1 = (1 - \frac{2GM}{r})\dot{t}^2 - \frac{\dot{r}^2}{(1-\frac{2GM}{r})} - r^2\dot{\phi}^2.$$

Now eliminate variables other than r, using the constants of motion we have from the above equations: $l/m = r^2\dot{\phi}$, and $E/m = (1 - \frac{2GM}{r})\dot{t}$, to get:

$$1 = \frac{E^2}{m^2(1-\frac{2GM}{r})} - \frac{\dot{r}^2}{(1-\frac{2GM}{r})} - \frac{l^2}{m^2r^2}.$$

We can write this in a nicer form by solving for $m\dot{r}^2$,

$$m\dot{r}^2 = \frac{E^2}{m} - \left(m + \frac{l^2}{mr^2}\right)(1 - \frac{2GM}{r}),$$

Remembering the definition of $\dot{r}$ and using dimensional analysis to put back the c's, we can write this as:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)\left(mc^2 + \frac{l^2}{mr^2}\right).$$

Notice that, as I mentioned in my handwaving description, a step in proper time $d\tau$ forces a step dr in the r direction. Thus this geodesic equation shows that things fall due to the spacetime curvature of the metric. We will look at these geodesic equations in some detail, but for now let's just take one limit of this last equation.

Suppose the angular momentum $l = 0$, which we might expect for radial infall towards a spherical mass. Our equation then is:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right)mc^2 = 0.$$

Next consider a case where you start at rest far from the object so that at proper time $\tau = 0$, $m(dr/d\tau)^2 = 0$, and $r \to \infty$. Our equation becomes:

$$\frac{E^2}{mc^2} - (1 - \frac{2GM}{\infty})mc^2 = 0,$$

or $E^2/mc^2 = mc^2$, or simply $E = mc^2$! So at $\tau = 0$ the total energy is just $E = mc^2$. In Newtonian mechanics the energy at infinity is usually defined as $E = 0$. Isn't it nice how these important results are just built right into the general relativistic view of spacetime. Also since energy E is conserved along geodesics we know that $E = m$ always. This E is not the Newtonian energy; it is *the* conserved quantity, which is better than the sum of $\frac{1}{2}mv^2 + V$. Finally note that if we would have started with some velocity at $r = \infty$ then $E > mc^2$ but it still would have been conserved.

At later times during this radial infall from rest, our equation becomes:

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - mc^2 + \frac{2GM}{rc^2}mc^2 = \frac{2GMm}{r}.$$

That is simply

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0,$$

which looks remarkably like the Newtonian case $\frac{1}{2}mv^2 - GMm/r = 0$! But this is not the same equation. It is fully relativistic and the time derivative is τ not t (remember $t = \gamma\tau$ in SR). In general you integrate these 4 equations to get the complete picture of motion near the Earth, Sun, or Black Hole. We will come back to these soon.