

# Physics 161: Black Holes: Lecture 7/8: 19/21 Jan 2011

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## 7 More Euler-Lagrange plus gravitational redshift

### 7.1 Conserved quantities in the Euler-Lagrange formalism: Energy and Momentum

Note that as we did the derivations of the geodesics above we came across quantities that did not change as we traced along the affine parameter, that is we found conserved quantities. This is a general and important thing to watch for in using the Euler-Lagrange formalism. In general there will be a conserved quantity whenever the Lagrangian  $L$  does not depend explicitly on one of the variables. Actually there will be one conserved quantity per variable. This is easy to see. To be general, let's use the 4-vector notation  $x_\mu$ , where  $x_0 = t$ ,  $x_1 = x$ ,  $x_2 = y$ , and  $x_3 = z$ , so  $\mu$  runs from 0 to 3 and  $x_\mu$  can represent any of the spacetime variables. In this notation, all the Euler-Lagrange equations can be written in one line:

$$\frac{\partial L}{\partial x_\mu} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}_\mu} = 0.$$

So if  $L$  does not depend on one variable, call it  $x_\mu$ , we have  $\frac{\partial L}{\partial x_\mu} = 0$ , and the Euler-Lagrange equation reads:  $\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}_\mu} = 0$ . Thus along the geodesic (running over values of the affine parameter  $\lambda$ ), the quantity  $\frac{\partial L}{\partial \dot{x}_\mu}$  is a constant.

In general we define  $p_\mu$ , the **conjugate momentum**, of a variable  $x_\mu$ , as

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}.$$

Thus we see that if the Lagrangian does not depend explicitly on a variable, then that variable's conjugate momentum is conserved. This  $p_\mu$  may be an actual momentum, but it could be some other conserved quantity. If one takes  $L = \frac{1}{2}m\dot{x}^2$ , then in fact  $p_x = mv_x$ , which is conserved since  $L$  does not depend upon  $x$ . If instead we took  $L = \frac{1}{2}m\dot{x}^2 - V(x)$ , then the conjugate momentum is still  $p_x = mv_x$ , but not it is not conserved along the geodesic as the affine parameter ( $t$ ) changes. Of course this is because there is a force due to the potential energy.

For the Minkowski metric, let's see what these conjugate momenta are. We saw above that the metric did not depend explicitly on any of the  $x_\mu$ ;  $t, x, y$ , or  $z$ , thus we expect to have 4 conserved quantities. Consider the  $t$  equation with affine parameter proper time  $\tau$ . We have  $p_t = \frac{\partial L}{\partial \dot{t}} = \dot{t}$ . Likewise for the  $x$

equation we find  $p_x = \dot{x}$ ,  $p_y = \dot{y}$ , etc. During the above calculation we found that the constant for the Euler-Lagrange equations (which we called  $c_t$ ) was equal to the Lorentz factor  $\gamma$ . Thus we see that

$$p_0 = p_t = \gamma = E/m,$$

where we noticed that in special relativity the energy of a particle is  $E = m\gamma$ , where  $m$  is the rest mass. Thus the conserved quantity associated with the  $t$  variable is nothing other than the energy. (It is energy per unit mass actually, but since  $p_t$  is a constant we can multiply by another constant  $m$  and still have a constant.)

We also previously found the momentum conjugate to the  $x$  variable,  $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} = c_x$ . Using  $c_x/c_t = v$  from before, we have  $p_x = v_x c_t = v_x p_t = v_x \gamma$ . But in special relativity the momentum is just  $mv\gamma$ , so see that  $p_x = P_x/m$ , is the momentum per unit mass. (We call the actual momentum  $P$  to distinguish it from the momentum per unit mass.) Thus the 4 conserved quantities we discovered are just the energy and 3 components of momentum.

We can find the relation between these quantities from our definition of the Lagrangian:  $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$ . Substituting in  $p_t = \dot{t} = E/m$ ,  $p_x = \dot{x} = P_x/m$ , etc. we find

$$1 = \sqrt{(E/m)^2 - (P_x/m)^2 - (P_y/m)^2 - (P_z/m)^2}.$$

Squaring, multiplying through by  $m^2$ , and using  $P^2 = P_x^2 + P_y^2 + P_z^2$ , we find the well known result  $E^2 = P^2 + m^2$ . Finally using units to put back the  $c$ 's,  $E^2 = P^2 c^2 + mc^4$ , which reduces to the famous  $E = mc^2$  in the limit of zero velocity (zero momentum).

## 7.2 Gravitational Redshift from the Schwarzschild metric

We already discussed the curvature of time in the Schwarzschild metric. Clocks run slower in a gravitation field (at smaller values of  $r$ ). This is one of the more measurable effects of GR. Measurements are possible because atomic transitions act like small clocks which send messengers (photons) to us here at Earth. Recall the proper time in the Schwarzschild metric:

$$d\tau = dt \sqrt{1 - \frac{2GM}{rc^2}},$$

where  $d\tau$  is tick of a clock measured at  $r$ , and  $dt$  is the length of that same tick measured far away ( $r \rightarrow \infty$ ).

Now consider some atomic transition, for example, the  $n = 3$  to  $n = 2$  transition of atomic Hydrogen. This is called the H-alpha transition and results in emission of a photon with energy  $E_\gamma = 13.6\text{eV}(\frac{1}{3^2} - \frac{1}{2^2}) = 1.89\text{ eV}$ . Using  $\lambda = c/\nu$ , and  $E_\gamma = h\nu$ , we find the corresponding wavelength  $H\alpha$  : 6563.5 **Angstroms**. Now in the frame at  $r$ , the photon has a fixed frequency  $\nu_0$ , and each oscillation is like a little clock ticking with period  $d\tau_0 = 1/\nu_0$ . Thus the time between crests in the outgoing photon measures  $d\tau$ . That same photon travels out to us at  $r = \infty$ . Then that same period is observed as  $dt$ . Thus the  $d\tau_\infty \equiv d\tau_{obs} = dt = d\tau_0/\sqrt{1 - 2GM/rc^2}$  is longer. Using the definitions above of  $\nu$  and  $\lambda$  we see that the frequency decreases by the same factor and the more easily observed wavelength is longer at when observed at infinity than the emitted wavelength at  $r$  by

$$\lambda_{obs} = \frac{\lambda_0}{\sqrt{1 - \frac{2GM}{rc^2}}} = \frac{\lambda_0}{\sqrt{1 - \frac{r_s}{r}}},$$

where we used the definition of the Schwarzschild radius  $r_S$ .

The redshift of an emission line is defined as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} = \left(1 - \frac{r_S}{r}\right)^{-1/2} - 1,$$

where  $\lambda_0$  is the emitted wavelength in the rest frame of the atom. Note for small values of  $2GM/rc^2 = r_S/r$ , this expression can be Taylor expanded as  $z \approx GM/rc^2 = \frac{1}{2}r_S/r$ .

This gravitational redshift is totally different from either the normal Doppler shift redshift caused by an atom moving away from the observer, or the cosmological redshift caused by the expanding Universe. Let's see how big the effect is by calculating it for some typical astronomical objects. First how about emission from the surface of the Sun? We expect the wavelengths received by us on Earth to be longer than the emitted wavelengths since the emitted photons have to climb out of the potential well of the Sun where clocks run slower. Using the approximation above  $z_{\odot} \approx \frac{1}{2}3\text{km}/7 \times 10^5\text{km} \approx 2 \times 10^{-6}$ . This is not a very big effect; a clock on the surface of the Sun loses about 30 seconds/year. For H-alpha the line redshift would be only 0.014 Angstroms.

How about for a white dwarf star? The mass of a white dwarf is about  $0.6M_{\odot}$ , and the radius is about 5500 km, near to the Earth radius. Thus  $z \approx \frac{1}{2}(.6)(3\text{km})/5500\text{km} \approx 1.6 \times 10^{-4}$ , and the wavelength of H-alpha emission received on Earth would be  $6563.5(1+z) = 6564.6$  Angstroms, quite easily distinguishable from local H-alpha lines. This result agrees well with the measurements from the nearby white dwarf star Sirius B. A clock on the surface of Sirius B would lose about 0.6 second/hour.

Next consider the case of a neutron star, with  $M = 1.4M_{\odot}$  and  $R = 10$  km. Now  $r_S = (1.4)(3\text{km}) = 4.2$  km, so really should not use the approximation, but calculate  $z = [1 - (4.2\text{km}/10\text{km})]^{-1/2} - 1 = 0.313$ . This is big shift: the H-alpha line would appear at 8618.3 Angstroms and clocks would run 31% slower.

Finally what about light emitted from the Schwarzschild radius of a black hole? With  $r = r_S$ ,  $z \rightarrow \infty$ ! Light would be invisible since it would be redshifted to infinitely low energies. Clocks sitting at that location would appear to have stopped!

## 8 Light Bending and Gravitational Redshift from the Equivalence Principle

To find proper answers in GR we calculate using the metric, in our case the Schwarzschild metric, but there is an interesting, more intuitive way that uses the Equivalence Principle directly. Let's look briefly at two examples.

### 8.1 Light bending

Imagine someone in a falling elevator. The equivalence principle says that experiments done in this falling frame will give the same results as experiments done floating in free space far from any massive object. Suppose the guy in the falling elevator shines a laser beam horizontally from one side of the elevator to the other. That light must follow geodesics and therefore go "straight" across the elevator; if the beam leaves from a height of 3 feet above the elevator floor, it must arrive on the other side also three feet above the floor. From the point of view of the elevator man, the light must travel exactly horizontally. But now think of someone watching through the (glass) walls of the elevator. The man shoots the laser from a height of 3 feet above the floor, but then while the light is traveling across, the elevator continues to fall. Thus when the light hits the opposite wall 3 feet above the floor, the wall has moved down. Therefore

the light cannot go straight in the Earth frame. The light must also fall. This gedanken experiment shows the light must fall in a gravitational field in the same way as everything else. Of course, from the point of view of GR, light is just following the geodesic through a curved spacetime; we will calculate those soon, but here we foresee what that calculation will give. In particular, light from distant stars travelling near the limb of the Sun must be “deflected” towards the Sun. Sir Author Eddington did this experiment in 1919, and it garnered world wide fame for Albert Einstein, because it was one of the first proofs of Einstein’s GR theory. In the homework, I am asking you to calculate how far the light falls in a gravitational field, that is how much it bends.

## 8.2 Gravitational Redshift again

The equivalence principle can also be used to show that light must gravitationally redshift. Again consider our man in a falling elevator. This time he shoots his laser from the floor to the ceiling.

First think about it from the point of view of someone outside watching through the glass walls. In the Earth frame the freely falling elevator picks up speed while the light is in transit, and so there is Doppler shift caused by the speeding elevator ceiling. The time the photons spend in transit is  $t = h/c$ , where  $h$  is the height of the elevator, and the speed increase of the elevator is  $\Delta v = gt = gh/c$ , where  $g = GM_{\text{Earth}}/r^2 = 9.8\text{m/s}^2$ . We could find the expected Doppler shift from the relativistically correct formula  $\nu_{obs} = \nu_0 \sqrt{(1 + \Delta v)/(1 - \Delta v)}$ , but a Taylor expansion of this is good enough for the slow speeds involved here:  $\nu_{obs} \approx \nu_0(1 + \frac{1}{2}\Delta v)(1 + \frac{1}{2}\Delta v) \approx \nu_0(1 + \Delta v)$ , or  $\Delta\nu = \nu_{obs} - \nu_0 = \Delta v\nu_0$ . Thus in the Earth frame we expect the light beam to be blue shifted by  $\Delta\nu/\nu_0 = \Delta v/c = gh/c^2$ , where I put the  $c$ 's back in. However, the equivalence principle says that this cannot be! The man in the elevator can't tell whether he is falling or floating freely in deep space, so he expects correctly that the frequency of light received at the elevator's ceiling is the same as the frequency he sent from the floor. Thus there must be another effect to cancel this one! The light traveling up must be gravitationally redshifted to counterbalance this effect. Thus there must be a factor of  $\Delta\nu/\nu_0 = -v/c = -gh/c^2 = -GMh/r^2c^2$ . This is not quite the result we got before from the metric, since there is a factor of  $r^{-2}$  rather than  $r^{-1}$ , but before we found the redshift between  $r = r_0$  and far away ( $r = \infty$ ) and here we found it only between  $r_0$  and  $r$ . So we can continue with this derivation by integrating the small redshift we found  $\Delta z \sim \Delta\nu/\nu_0$  from  $r_0$  to  $\infty$ .

$$z \sim \int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \approx \int_{r_0}^{\infty} -\frac{GM}{r^2c^2} dr.$$

This gives  $\ln(\nu_\infty/\nu_0) \approx -GM/r_0c^2$ , or  $\lambda_\infty = \lambda_0 \exp(GM/r_0c^2)$ . This is still not the correct answer, but if we assume the factor in the exponential is small and Taylor expand the exponential using  $e^x \approx 1 + x + x^2/2 + \dots$ , we do find  $\lambda_\infty/\lambda_0 \approx 1 + GM/r_0c^2$ , or  $z = GM/r_0c^2 = \frac{1}{2}r_S/r$ , the Schwarzschild result for small  $z$ .