

Physics 161: Black Holes: Lecture 20: 18 Feb 2011

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20 Inside the Black Hole: Kruskal-Szerkeres coordinates

20.1 Coordinate problem at $r = r_S$

Let's look at the point $r = r_S = 2GM/c^2$ in the Schwarzschild metric. The metric blows up here, but we found that for things falling in, there is no problem. Forces don't become large, in fact nothing too bad happens. The problem here is not the physics, it is the coordinates. There is a coordinate singularity at $r = r_S$, but no physical singularity. Besides, coordinates where t runs backward, and r and t switch roles are just confusing.

As a first example of how coordinates can be bad consider polar coordinates on a plane:

$$ds^2 = dr^2 + r^2 d\theta^2.$$

In the angular direction this says the distance is $ds_\theta = r d\theta$. But what look what happens as $r \rightarrow 0$. Then $ds/r = d\theta \rightarrow \infty$, and the metric doesn't make any sense.

Fig: Grid of coordinates on surface of sphere

As a second example of coordinate problems, consider the surface of the sphere.

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2,$$

where R is a constant (the radius of the sphere). We can plot out these coordinates on a grid ϕ vs. θ and everything looks fine. However, at $\theta = 0$, all the values of ϕ are the same point, that is ϕ is not defined. However, there is nothing special about the actual point $\theta = 0$ on a sphere. It is only the coordinates that are bad at the point $\theta = 0$. In fact it is very easy to fix this problem by just moving the origin to another point on the sphere!

The problem with the Schwarzschild metric at $r = r_S$ is similar; it is not a physical singularity, but only a coordinate singularity. Remember how light cones squeeze up and then flip sideways as one crosses r_S ?

Fig: Coordinate singularity of Schwarzschild coordinates

20.2 Kruskal-Szerkeres coordinates

Knowing the problem above for the Schwarzschild coordinates, there was a long hard search for better coordinates to describe the inside and horizon of black holes. These were found in 1960 by Kruskal and

Szekeres. These coordinates show the complete geometry inside the black hole and hold some surprises. Changing coordinates is just changing variables. Here are the definitions: For $r > r_S$ (outside black hole):

$$u = \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/2r_S} \cosh \frac{ct}{2r_S},$$

$$v = \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/2r_S} \sinh \frac{ct}{2r_S},$$

while for $r < r_S$ (inside black hole):

$$u = \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/2r_S} \sinh \frac{ct}{2r_S},$$

$$v = \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/2r_S} \cosh \frac{ct}{2r_S}.$$

With this change of variable the Schwarzschild metric becomes:

$$ds^2 = \frac{4r_S^3}{r} e^{-r/r_S} (-dv^2 + du^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

It is easiest to see this by calculating du and dv in terms of dt and dr and plugging these into the equation above to recover the Schwarzschild metric. Recall the definitions of the hyperbolic sine and cosine: $\cosh x = (e^x + e^{-x})/2$, and $\sinh x = (e^x - e^{-x})/2$, with identity: $\cosh^2 x - \sinh^2 x = 1$. One sees immediately that this metric does not contain any singularity or problem at $r = r_S$. It does however have a problem at $r = 0$. This is true physical singularity. The density and curvature become infinite there, so no change of coordinates can remove that problem. This is a problem for quantum gravity or string theory.

For use later we note the following identity (valid both inside and outside the black hole):

$$u^2 - v^2 = \left(\frac{r}{r_S} - 1\right) e^{r/r_S}.$$

Let's plot out and explore this form of the Schwarzschild metric. We plot u vs. v , and try to understand how particles move in these new coordinates. Everything we did before is still valid, it is just the coordinates that are different; like switching from Cartesian to Polar coordinates on a plane.

Recall that we find the time and space coordinates by the signs of the terms in the metric. We see that θ , ϕ , and u are space-like coordinates, and that v is timelike. So we plot u on the horizontal axis and v on the vertical axis.

Fig: Kruskal-Szekeres coordinates

- First note that light travels at 45° everywhere in these coordinates. Remember we find the null (lightlike) geodesics by setting $ds = 0$. We can also set $d\theta = d\phi = 0$ to consider only radial moving light. The KS metric then reads: $0 = -dv^2 + du^2$, or $dv/du = \pm 1$. Thus lightcones look exactly like they do in flat Minkowski space. This is the best property of these coordinates. There is no squeezing up as one approaches the horizon, or flipping over inside the black hole.

- Regions of constant r are hyperbolas (i.e. $u^2 - v^2 = \text{constant}$). Thus particles that don't move at all in the Schwarzschild metric actually move along hyperbolas in these coordinates. This can be seen from the above equation for $u^2 - v^2$ which depends only on r and not on t . So different values of constant r give different hyperbolas. The old variable t advances as you move along one of these hyperbolas. A particle at rest does not move straight up in u and v coordinates, but moves along a hyperbola.
- Note that the singularity point $r = 0$ is also a hyperbola. Plug in $r = 0$ and find $u = \sinh ct/2r_S$ and $v = \cosh ct/2r_S$. Noting that $\cosh^2 x - \sinh^2 x = 1$, we see that $v^2 - u^2 = 1$, is the hyperbola of the singularity at the center of a black hole. Thus in these coordinates the singularity is spread out over u and v . In fact, there are two hyperbolas here. These coordinates show a better picture of the singularity than the Schwarzschild coordinates. We draw singularities as jagged lines; this indicates nothing can pass by and anything that touches will be destroyed. In our plot we have suppressed the theta and phi directions. Really then the singularity is a set of two surfaces! (Rotate our coordinate around the axis to represent say the phi direction.
- To find where lines of constant t are, divide the definitions of u and v . We see that $v/u = \tanh ct/2r_S$ outside the black hole and $v/u = \coth ct/2r_S$ inside. Thus v/u depends only t and not on r , so surfaces that have constant t have constant values of v/u , and actually are straight lines. We can draw these on our coordinate plot. Where these lines cross the hyperbola shows us how time evolves along the hyperbolas. We see that lines with higher slopes represent later times. Actually we knew this already because light travels up at a 45° angle toward the future.
- The line $t = \infty$ is the line $u = v$, that is, $\tanh(ct/2r_S) = \pm 1$ has as its solution, $t = \pm\infty$. Remembering that lightcones are still 45° angles one can go from a finite time with $v < u$ past $t = \infty$ to where $v > u$. However, after $t = \infty$, we see that all future lightcones will lead to the hyperbola that represents the singularity. Thus the inside of the black hole in these coordinates is the region above the diagonal lines, and one still has to hit the singularity and get killed once you are inside the black hole.
- We see that the horizon of the black hole is thus the null line that goes through the origin. These 45° lines divide up the entire spacetime into four regions. The outside to the right is called region I. It is where we start when we fall into the black hole.
 The region on the top is called region II. It contains the inside of the black hole. The singularity towards the future is called the black hole.
 The region to the left is called region III. It is another asymptotically flat region. This region is connected to region I by the wormhole.
 The region on the bottom is called region IV. The singularity on the bottom (towards the past) is called the white hole. Note that the future lightcones show that it is impossible to ever reach the white hole.
- Using the identity above for $u^2 - v^2$, we see that inside the hole lines of constant r are side-to-side hyperbolas like the singularity, but outside the hole lines of constant r are up-and-down hyperbolas. A shell observer outside the hole therefore moves along one of these up-and-down hyperbolas. The world lines of objects at rest are not straight up and the slope of the world lines does not give the velocity. However, moving at c is a 45° angle. So if you drop an object into the hole from a shell it would start to accelerate towards the hole, that is, the slope of its world line would shift towards

the hole and eventually approach 45° as it crossed the horizon. Once inside it would eventually hit the singularity on the top.

So we see that a black hole geometry is more complicated than Schwarzschild imagined. We recognize regions I and II as the normal outside and inside of a black hole, but what are these extra regions? Region III is pretty interesting. If you go far along the $-u$ axis you get to a place where spacetime is flat. This is called asymptotically flat which is like the far-away coordinates we are used to, except this region III is not just the same outside as region I. It seems to be a completely new area. In theory it could be a completely different Universe, or a part of this Universe far from region I. Thus black holes seem to connect different regions. This can be shown using an embedding diagram.

Fig: Schwarzschild wormhole

The area of the diagram between regions I and III is called the **Einstein-Rosen bridge, or Schwarzschild throat or Schwarzschild wormhole**. If you could go through it you could enter the black hole from region I, and come out of the black hole into region III, which could be millions of lightyears away, or even part of another Universe!

However, can you get through the wormhole from region I to region III? Try and do it. Start from anywhere in region I and stay inside the future lightcone. You see that because of the 45° angle of the lightcone, you can't go from region I to region III. The closest you can come is to start at the $t = -\infty$ null line, but even in that case you can't get out of region II and will hit the singularity. So Schwarzschild wormholes can't be used to get anywhere. We will talk about other wormholes later.

How about region IV, the white hole. Well, things can come out of white holes, but nothing can go in. They only exist in the past lightcones. In fact, the same theorems that say that black holes can't be destroyed also say that white holes can't be created. So unless they were formed at the beginning of the Universe, we don't expect to find any.

Finally, we note that in real black holes we don't expect regions III or IV to exist. We see this by imagining the creation of a black hole due to the collapse of a star. Remember the Schwarzschild metric is valid only outside of spherical objects. Inside objects, one would need a different solution to Einstein's equations. Thus follow the collapse of a star into a black hole. The surface of the star follows a path just like when we dropped a particle into the hole. But only outside this surface does the Schwarzschild metric hold. As the surface collapses it moves up, finally entering region II, which is when the star turns into a black hole. At that point the trapped surface forms, and all the material in the star gets crushed into the singularity. But all this time regions III and IV were not really there. So in the final black hole we find no reason to suppose that these regions exist. They are mathematical solutions to Einstein's equations, but don't exist because they were never formed.