

Physics 161: Black Holes: Lecture 5: 12 Jan 2011

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5 More Special Relativity; Gravitational Time dilation

5.1 Time dilation and length contraction by spacetime diagram

Last time we saw how to calibrate spacetime diagrams. We can use that result to derive the famous time dilation: moving clocks run slow to stationary observers.

Numerically we can calculate for time dilation for event B, considering starting a clock at the origin (event A). $\Delta s^2 = -\Delta t'^2 + \Delta x'^2 = -\Delta t^2 + \Delta x^2$. For a clock tick at event B, $\Delta t = t$, $\Delta x = x$, $\Delta t' = t'$. In the O , lab frame: Event B occurs $x = \Delta x = v\Delta t$, and time occurs at $t = \Delta t$. In O' , the moving frame, $\Delta x' = x' = 0$ since events B and A occur at the same place (O' is not moving in its own frame). So $-\Delta t'^2 + 0 = -\Delta t^2 + \Delta t^2 v^2 = -(1 - v^2)\Delta t^2$, or $\Delta t = \Delta t' / \sqrt{1 - v^2} = \gamma \Delta t'$. This is the time dilation you learned before in special relativity.

One can similarly find out the length contraction you learned about in SR. If you remember the derivation, you will remember that it is more tricky than the time dilation calculation. The reason for the trickiness is easy to see in the spacetime diagram. Consider the origin as point A and a ruler of length a laid out on a spacecraft with its far end at point B. How does that look? It lies along the x' axis, which we have just seen how to find (the set of points which all have $t' = 0$). Now draw the sideways hyperbola of all proper distances a . If we just project the point B onto the x axis we find the wrong answer, that the ruler looks *longer* in the lab frame. What have we done wrong? By length of ruler in the lab frame we mean the distance between the two ends measured at the same lab frame time t . Thus the distance between points A and B is not the distance we are after. We need to follow point A up in time until it reaches the same time t as point B has. We know the t' axis has the same angle from the 45° line as the A-B segment, so we can just go along the t' axis until we get as high as point B. Now the length of the ruler as measured in the lab frame is the difference in x of the two ends measured at the same time. We see the ruler is shorter as measured in the lab frame as expected.

5.2 Other special relativity you need to know

This class has special relativity as a prerequisite, so I'm not going to cover much more. However, you should look over the book to refamiliarize yourself with several topics. For example, you should understand $E = mc^2$. In fact, you need to know that the real equation is $E^2 = (mc^2)^2 + (pc)^2$, where p is the momentum. With $c = 1$ this is more suggestively written: $m^2 = E^2 - p^2$, and we see that the rest mass is an invariant in much the same way that the interval ds is. You also should be familiar with relativistic energy and momentum: $E = m\gamma c^2$, and $p = mv\gamma$. Try Taylor expanding the relativistic

expression for kinetic energy $T = E - mc^2 \approx \frac{1}{2}mv^2$, and find the next order correction to the famous Newtonian formula.

5.3 Time Dilation in a gravitational field

Remember the metric of flat spacetime is

$$ds^2 = -c^2 dt^2 + dx^2,$$

where we have suppressed the y and z dimensions. In this metric, we find the proper time by setting $dx = 0$ and using $d\tau = \sqrt{-ds^2}/c = dt$, and see that coordinate t is actually the proper time between events. Likewise the proper distance is $dl = dx$, so that the coordinate x is the proper distance. Compare this to the the Schwarzschild metric

$$ds^2 = -c^2 \left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

Here the proper time is found by setting $dr = d\theta = d\phi = 0$. Note then $ds^2 \leq 0$, that is a time-like separation. The proper time is again

$$d\tau = \sqrt{-ds^2}/c = \sqrt{1 - \frac{2GM}{rc^2}} dt.$$

So if $dt = 1$ sec, $d\tau = \sqrt{1 - \frac{2GM}{rc^2}}$ sec, which is a smaller (i.e. shorter) time. This proper time is the clock time measured at a distance r from the Earth (or a black hole). Thus we see that clocks run slower in a gravitational field than in deep space (where $r \rightarrow \infty$ and $dt = d\tau$).

Let's see how big effect this is for us living on the surface of the Earth. $M_{\text{Earth}} = 6 \times 10^{27}$ gm, and $r_{\text{Earth}} = 6.38 \times 10^8$ cm, and a useful number to calculate is the Schwarzschild radius of the Earth $r_{SC} = 2GM_{\text{Earth}}/c^2 = 0.886$ cm. Then $2GM_{\text{Earth}}/r_{\text{Earth}}c^2 = .886/6.38 \times 10^8 = 1.4 \times 10^{-9}$. Using a Taylor expansion, this says $d\tau = \sqrt{1 - 1.4 \times 10^{-9}} \approx 1 - \frac{1}{2}1.4 \times 10^{-9}$. Or $d\tau = (1 - 6.9 \times 10^{-10})dt$. So time runs slower here on Earth (as measured from deep space by about 1 sec every 45 years! Not a big effect, but measurable. Gravity curves time.

However in other environments this factor can be bigger. Especially wierd is what it does when $r = r_{SC}$, Then $\sqrt{1 - 2GM/rc^2} \rightarrow 0$, and the measured time seems to stop! This occurs at the horizon of a black hole.

5.4 Old Idea of Black Holes

The idea of black holes has been around since the 1700's when Laplace, Mitchell, and others thought about giant stars. One can even derive the Schwarzschild radius using Newton's laws and get the right answer (by coincidence I think?) Consider a big spherical ball of mass M and radius r . The gravitational potential energy at the surface of the ball is $V = -GMm/r$, where m is just some small test mass. Since V goes to zero at infinity, the **escape velocity** from the surface is found by setting this potential energy to the the kinetic energy $E = \frac{1}{2}mv^2$, and solving for v . This gives $v_{esc} = \sqrt{2GM/r}$. This escape velocity increases as the mass increases, or the radius decreases. And one could ask what happens when the escape velocity is equal to or larger than c . Then according to Newton's corpuscular theory of light (which is incorrect!) light would get trapped in such an object, and the giant star would be dark. Setting

$v_{esc} = c$ one finds the radius of such an object is $r = 2GM/c^2$, precisely the Schwarzschild radius. This Newtonian calculation gives the right answer, but is wrong. For example, light particles are massless so $E \neq \frac{1}{2}mv^2$. There are several errors in this calculation which just happen to cancel. The correct description requires the Schwarzschild metric.