Physics 4E **Final Exam Solutions** Spring 2011

Problem 1

(a) Wien's law: $\lambda_m T = \frac{hc}{4.96\, k_B} \Rightarrow T = \frac{hc}{4.96\, \lambda_m k_B} \Rightarrow$

$$\Rightarrow T = \frac{12{,}400}{4.96 \times 4000} \times 11{,}600\ ^\circ K \Rightarrow \boxed{T = 7250\ ^\circ K}$$

(b) $\boxed{E = \frac{hc}{\lambda} = 3.1\,eV}$ for $4000\,\mathring{A}$. $\boxed{\text{For } 40{,}000\,\mathring{A},\ E = 0.31\,eV}$

(c)
$$\frac{n_{photons}(\lambda = 40{,}000\,\mathring{A})}{n_{photons}(\lambda = 4000\,\mathring{A})} = \frac{e^{\frac{hc}{4000\mathring{A} kT}} - 1}{e^{\frac{hc}{40{,}000\mathring{A} kT}} - 1} = \frac{141.6}{0.642} = 220$$

There are 220 times as many photons in a mode of wavelength 40,000 Å as there are in a mode of wavelength 4000 Å.

(d) Because there are many more modes at wavelength 4000 Å than at 40,000 Å. The number of modes per unit volume is

$$n(\lambda) = \frac{8\pi}{\lambda^4} \Rightarrow \frac{n(4000\,\mathring{A})}{n(40000\,\mathring{A})} = 10^4 = \boxed{10{,}000}$$

The power is proportional to the number of modes, the energy of the photon in that mode, and the average number of photons in that mode. So:

$$\frac{\text{power}(4000\,\mathring{A})}{\text{power}(40{,}000\,\mathring{A})} = \frac{n(4000\,\mathring{A})}{n(40000\,\mathring{A})} \times \frac{n_{photons}(4000\,\mathring{A})}{n_{photons}(40{,}000\,\mathring{A})} \cdot \frac{E(4000\,\mathring{A})}{E(40000\,\mathring{A})} =$$

$$= 10{,}000 \cdot \frac{1}{220} \cdot 10 = \boxed{454}$$

Problem 2

$$E_n = -E_0 \frac{Z^2}{n^2} \quad , E_0 = -13.6\, eV \quad ; \quad hc = 12{,}400\, eV\, \mathring{A}$$

At room temperature, electrons are in the ground state.

The shortest wavelength absorbed corresponds to: $n=1$ to $n \to \infty$, so

$$\Delta E = E_0 Z^2 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_0 Z^2} = \frac{911.76\ \mathring{A}}{Z^2}$$

So Z can't be 1 nor 2, nor ≥ 4. $\boxed{Z = 3}$ $\boxed{\frac{hc}{E_0 Z^2} = 101.307\, \mathring{A}}$

(e.g. for $Z=4$, shortest wavelength absorbed is $\frac{911.76}{16}\mathring{A} = 57\mathring{A}$, longest is $\frac{911.76}{Z^2} \cdot \frac{4}{3} = 76\mathring{A}$)

For $Z=3$:

$n=1 \to n=2$: $\Delta E = E_0 Z^2 \cdot \frac{3}{4} = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_0 Z^2} \cdot \frac{4}{3} = \boxed{135.08\, \mathring{A}}$

absorption

$n=1 \to n=3$: $\Delta E = E_0 Z^2 \left(1 - \frac{1}{9}\right) = \frac{8}{9} E_0 Z^2 = \frac{hc}{\lambda} \Rightarrow$

$\Rightarrow \boxed{\lambda = \frac{hc}{E_0 Z^2} \cdot \frac{9}{8} = 113.97\, \mathring{A}}$ absorption

For emission: $n=3 \to n=2$, $\lambda = \frac{hc}{E_0 Z^2} \cdot \frac{36}{5} = 729.4\, \mathring{A}$

Emission lines: $\boxed{\lambda = 135.08\mathring{A},\ \lambda = 113.97\mathring{A},\ \lambda = 729.4\mathring{A}}$

(c) $L = m_e v r = \hbar \quad , r = \frac{a_0}{Z} \Rightarrow v = \frac{\hbar}{m_e a_0} Z \Rightarrow$

$$\Rightarrow \frac{v}{c} = \frac{\hbar c}{m_e c^2 a_0} Z = \frac{1973}{0.511 \times 10^6 \times 0.529} \times 3 = 0.0219$$

$$\boxed{\frac{v}{c} = 0.0219}$$

Problem 3

(a) $E_n = \hbar\omega(n+\frac{1}{2})$ for harmonic oscillator

$\omega = \sqrt{\frac{K}{m}}$. So $\frac{\omega_{electron}}{\omega_{neutron}} = \sqrt{\frac{m_{neutron}}{m_{electron}}} = \sqrt{1849} = 43$

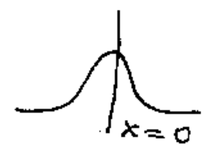
So electron is in state $n=0$, neutron is in state n with same energy

$\Rightarrow \hbar\omega_{el} \cdot \frac{1}{2} = \hbar\omega_{ne}(n+\frac{1}{2}) = \hbar \cdot \frac{\omega_{el}}{43}(n+\frac{1}{2}) \Rightarrow$

$\Rightarrow 43\,\cancel{\omega_{el}} \cdot \frac{1}{2} = \cancel{\omega_{el}}(n+\frac{1}{2}) \Rightarrow \boxed{n=21}$

So there are 21 states for the neutron with energy smaller, namely

$\boxed{n=0, n=1, \ldots, n=20.}$

(b) The ground state wavefunction for electron looks like 

so it is $\boxed{\text{most likely to be found at } x=0.}$

The neutron is in a highly excited state which approaches classical behavior, so it is most likely found at the classical amplitude of oscillation for this energy, namely $\boxed{x = \pm 1\,\text{Å}.}$

(c) For neutron, $E_0^{neu} = \hbar\omega_{neu} \cdot \frac{1}{2} = \hbar\omega_{el} \cdot \frac{1}{2} \cdot \frac{\omega_{ne}}{\omega_{el}} = \frac{E_0^{el}}{43}$

So $\boxed{E_0^{neu} / E_0^{el} = 1/43}$

(d) Electron has higher uncertainty in position

$\text{Energy} = \frac{1}{2} K A^2 \Rightarrow \frac{E_{el}}{E_{neu}} = 43 = \frac{A_{el}^2}{A_{neu}^2} \Rightarrow A_{neu} = \left(\sqrt{43}\right)^{-1} A_{el}$

$\Rightarrow \Delta X_{neu} = 6.56^{-1} \Delta X_{el} \Rightarrow \boxed{\Delta X_{neu} = 0.1525\, \Delta X_{el}}$

Problem 4

$$E_n = \frac{\hbar^2 \pi^2}{2 m_e L^2} n^2 = \frac{3.81 \text{ eV} \mathring{A}^2 \pi^2}{9 \mathring{A}^2} = 4.178 \text{ eV} \cdot n^2$$

$$\boxed{E_1 = 4.178 \text{ eV}, \quad E_2 = 16.71 \text{ eV}}$$

The exact answers will be smaller, particularly for E_2, because the wavefunction penetrates the forbidden region where $V(x) = V_0$, so Δx increases and the energy decreases. The penetration is larger for the higher energy because $\Psi \sim e^{-\alpha x}$, $\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

(b) The tunneling probability per attempt is:

$$T = e^{-2\alpha \Delta x}, \quad \Delta x = 0.8 \mathring{A}, \quad \alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \Rightarrow$$

$$\alpha_1 = \sqrt{\frac{1}{3.81}(20 - 4.178)} \; \mathring{A}^{-1} = 2.038 \mathring{A}^{-1}, \quad \alpha_2 = \sqrt{\frac{1}{3.81}(20 - 16.71)} = 0.929 \mathring{A}^{-1}$$

$$\Rightarrow T_1 = e^{-2\alpha_1 \cdot \Delta x} = 0.038, \quad T_2 = e^{-2\alpha_2 \Delta x} = 0.226, \quad \boxed{\frac{T_2}{T_1} = 5.95}$$

The speed of the electron is given by $\frac{1}{2} m_e v_n^2 = E_n = \frac{\hbar^2 \pi^2}{2 m_e L^2} n^2$.

Therefore, the ratio of the speeds is $\frac{v_2}{v_1} = 2$

The attempt frequency $=$ 1 / time it takes to travel across well twice $= v / 2L$

$\frac{\text{Tunneling probability}}{\text{unit time}}$ = tunneling probability per attempt $\times$ attempt frequency $= \frac{v \cdot T}{2L}$

We have $\frac{v_2 \cdot T_2}{v_1 T_1} = 2 \times 5.95 = 11.9$. The time it takes to escape is inversely proportional to the tunneling probability per unit time

$\Rightarrow$ $\boxed{\text{time to escape from } n=1 = 11.9 \times \text{time to escape from } n=2 = 11.9\, t_0}$

Problem 5

$$\psi(r,\theta,\phi) = C r^2 e^{-r/a_0} \sin\theta \cos\theta \, e^{-2i\phi}$$

since ϕ dependence is $e^{+i m_\ell \phi}$ $\Rightarrow$ $\boxed{m_\ell = -2}$ $\Rightarrow$ $\ell \geq 2$

since r dependence is e^{-Zr/na_0} $\Rightarrow$ $Z = n$; $\ell \geq 2 \Rightarrow n \geq 3$.

the factor r^2 indicates that $n = 3$, $\ell = 2$ $\Rightarrow$ $Z = 3$. this is also consistent with the θ dependence (polynomial of degree ℓ in $\sin\theta$, $\cos\theta$)

hence $\boxed{n = 3,\ \ell = 2,\ m_\ell = -2,\ Z = 3}$

(b) Radial probability is $P(r) = r^2 R^2(r)$

$R(r) = \text{const} \times r^2 e^{-r/a_0}$, we don't need to know the const.

$$P(r) \propto r^6 e^{-2r/a_0} \Rightarrow P'(r) \propto 6r^5 e^{-2r/a_0} - \frac{2r^6}{a_0} e^{-2r/a_0} = 0$$

$\Rightarrow$ $r = \frac{6}{2} a_0$ $\Rightarrow$ $\boxed{r = 3a_0}$ most probable r.

(c) Bohr atom radius $r_n = r_0 n^2 = \frac{a_0}{Z} n^2 = \frac{a_0}{3} 3^2 = \boxed{3a_0}$

agrees with (b). It always does for $\ell = n-1$ which is the case here ($\ell = 2, n = 3$)

(d) Orbital magnetic moment: $\mu_z = -\frac{e}{2m_e} L_z$, $L_z = \hbar m_\ell \Rightarrow \mu_z = -\frac{e\hbar}{2m_e} m_\ell$

$\Rightarrow$ $\mu_z = -\mu_B m_\ell$. Energy change $\Delta E = -\mu_z B_z$ $\Rightarrow$

$\Rightarrow$ $\Delta E = +\mu_B m_\ell B_z$. For $m_\ell = -2$, $\Delta E < 0$ $\Rightarrow$ $\boxed{\text{energy decreases.}}$

$$\Delta E = -2\mu_B B_z = -2 \cdot 5.79 \times 10^{-5} \frac{eV}{T} \cdot 7T = -8.11 \times 10^{-4} eV$$

$$\boxed{\Delta E = -8.11 \times 10^{-4} eV}$$

(e) $\boxed{\text{If } B \text{ is pointing in the } x \text{ direction, } \Delta E = 0 \text{ since the } x\text{-component of } \mu \text{ is equally likely to be positive and negative.}}$

Problem 6

(a) At temperature much higher than the characteristic temperature T_R, the behavior is classical and the equipartition theorem holds. For rotation there are 2 degrees of freedom, each contributes $\frac{1}{2}kT$ to the average energy, hence $\boxed{\langle E_R \rangle = kT = 10\,kT_R}$

(b) At temperature $T \ll T_R$ the classical approximation doesn't work.

$$E_R = \hbar^2 \frac{L^2}{2I} = \frac{\hbar^2 \ell(\ell+1)}{2I} = k T_R\, \ell(\ell+1)$$

$E_R(\ell=0) = 0$, $E_R(\ell=1) = 2kT_R$. The probability that the molecule is in a state with energy E_R is proportional to $e^{-E_R/kT}$.

So

$$\langle E_R \rangle = \frac{\sum_\ell g_\ell E_R(\ell)\, e^{-E_R(\ell)/kT}}{\sum_\ell g_\ell e^{-E_R(\ell)/kT}} \cong \frac{g_1 E_R(\ell=1)\, e^{-E_R(\ell=1)/kT}}{g_0 + g_1 e^{-E_R(\ell=1)/kT}}$$

We have $g_\ell = 2\ell+1$, hence $g_0 = 1$, $g_1 = 3$ .. g_ℓ is the degeneracy of state ℓ.

Using also that $e^{-E_R(\ell=1)/kT} \ll 1$ for $T = 0.1\,T_R$,

$$\langle E_R \rangle = \frac{3 \cdot kT_R \cdot 2 \cdot e^{-2T_R/T}}{1 + \dots(\text{small})} = 6\,kT_R\, e^{-20} = \boxed{1.24 \times 10^{-8}\, kT_R}$$

(c) $\text{prob(ground state)} \propto e^{-0/k_BT} = 1$, $\text{prob(first excited state)} = 3\,e^{-2T_R/T}$

$$\Rightarrow \quad 1 = 3\,e^{-2T_R/T} \Rightarrow e^{2T_R/T} = 3 \Rightarrow 2\frac{T_R}{T} = 1.099 \Rightarrow T = \frac{2T_R}{1.099}$$

$$\Rightarrow \boxed{T = 1.82\, T_R}$$

Problem 4

The energy of a particle of mass m in an infinite well of length L is:

$$E = \frac{\hbar^2 \pi^2}{2mL^2} n^2$$

If the particle is an electron, $\frac{\hbar^2}{2m_e} = 3.81 \text{ eV} \mathring{A}^2$. $\frac{\hbar^2\pi^2}{2m_e} = 37.60 \text{ eV} \mathring{A}^2$.

For the narrow well of width $0.1 \mathring{A}$, the ground state energy has to be less than 40 eV so that it fits into this well:

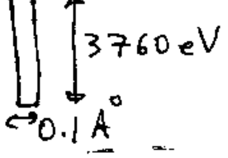
$$\Rightarrow \quad \frac{\hbar^2\pi^2}{2m(0.1\mathring{A})^2} < 40 \text{ eV} \Rightarrow \frac{m}{m_e} > \frac{\hbar^2\pi^2}{40 \cdot 2m_e(0.1\mathring{A})^2} = \frac{3760}{40} = 94 \sim 100$$

So $\boxed{m \gtrsim 100 m_e \cong 50 \text{ MeV}}$

(b) For an electron, we can ignore the narrow well. The ground state energy in the wide well: 37.60 eV, so measured from the bottom of the narrow well it is $\boxed{77.60 \text{ eV}}$

(c) By the uncertainty principle, $\Delta p \sim \frac{\hbar}{\Delta x}$, if Δx is so small, Δp is large $\Rightarrow$ the energy $E = \frac{(\Delta p)^2}{2m_e}$ is much larger than the depth of the well so the electron will not stay inside this well

For $L = 0.1\mathring{A}$, electron has minimum energy $E_1 = \frac{\hbar^2\pi^2}{2mL^2} = 3760 \text{ eV} \Rightarrow$ the narrow well needs to be at least as deep.



(d) This will be the case when the energy of the particle in the narrow well is approximately at the top of the narrow well, i.e. 40 eV, hence $m \sim 100 m_e$.