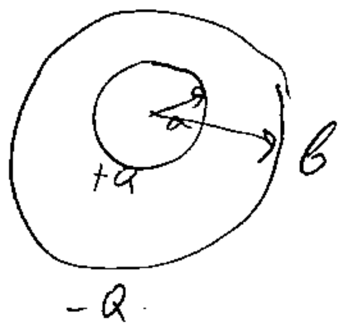
Homework 4

4.1A



The electric field inside is found from Gauss' law

$$E \cdot 2\pi r \ell = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{2\pi r \ell \varepsilon_0}$$

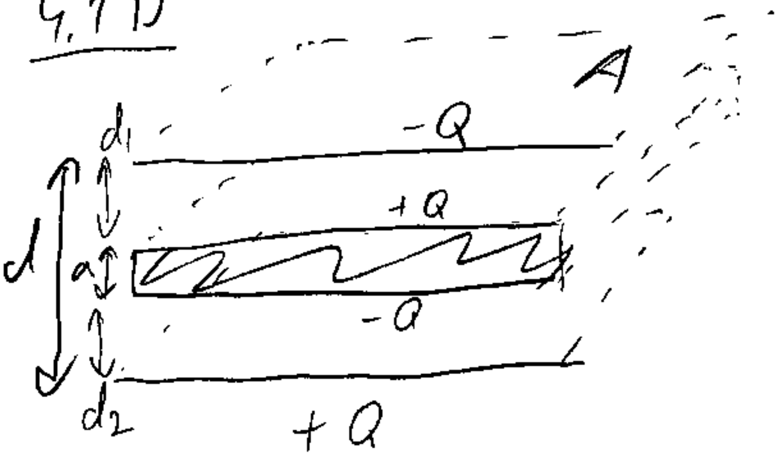
The voltage is

$$\Delta V = V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{r} = -\int_b^a \frac{Q}{2\pi r \ell \varepsilon_0} dr =$$

$$= -\frac{Q}{2\pi \varepsilon_0 \ell} \ln r \Big|_b^a = \frac{Q}{2\pi \varepsilon_0 \ell} \ln \frac{b}{a}$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi \varepsilon_0 \ell}{\ln \frac{b}{a}}$$

4.1 D



$$C_1 = \frac{\varepsilon_0 A}{d}$$

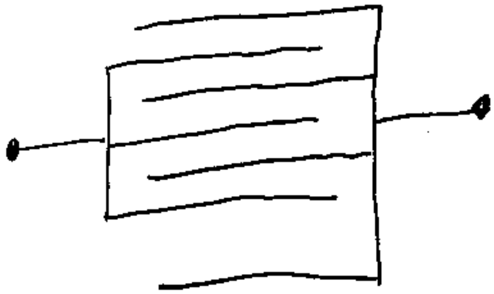
When the metal slab is inserted charges are induced on the surfaces as shown above. This means that there will be the same electric field inside as before, but only in the empty region, so effectively d gets reduced to $d-a$, which implies

$$C_2 = \frac{\varepsilon_0 A}{d-a} = \frac{\varepsilon_0 A}{d} \frac{d}{d-a} = \underline{C_1 \frac{d}{d-a}}$$

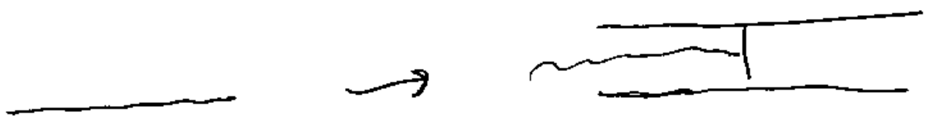
Another way of getting the same result is to consider the system as two capacitors in series:

$$\frac{1}{C_2} = \frac{1}{\varepsilon_0 A/d_1} + \frac{1}{\varepsilon_0 A/d_2} = \frac{d_1}{\varepsilon_0 A} + \frac{d_2}{\varepsilon_0 A} = \frac{d-a}{\varepsilon_0 A} \checkmark$$

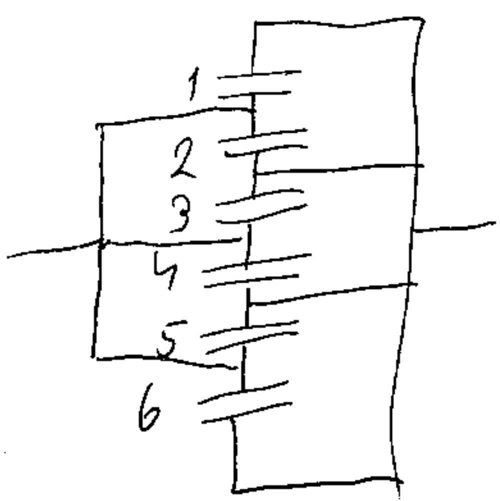
4.1 E



Consider each pair of plates as a capacitor (there is 6 overall). To do this the middle plates need to be replaced by a pair



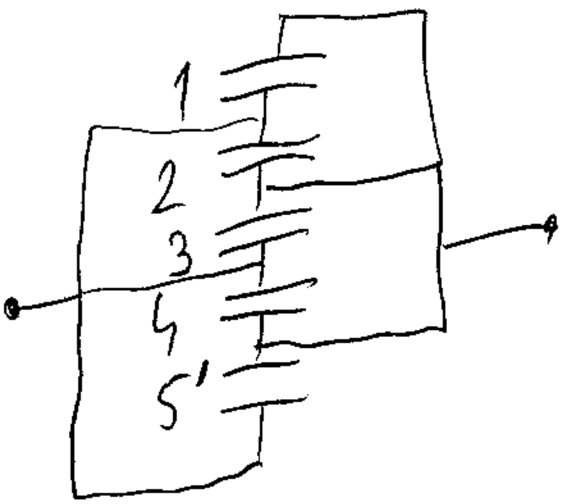
The above picture can then be drawn schematically:



Each capacitor has capacitance

$$C_1 = \frac{\varepsilon_0 A}{d}$$

It, is now not hard to see that 5 and 6 are in parallel and can be replaced by one capacitor 5' of capacitance $2C_1$:

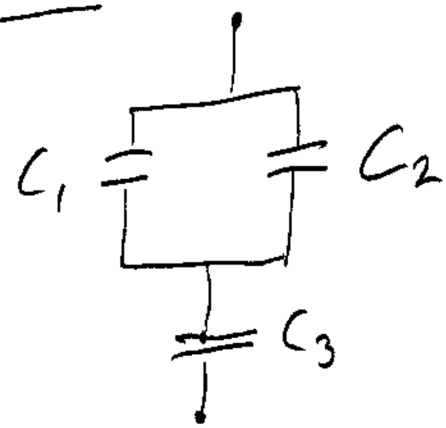


Now 5' is parallel to 4, and so on. In fact, all of them are parallel, so the overall capacitance is

$$C = 6C_1$$

$$\boxed{C = 6\frac{\varepsilon_0 A}{d}}$$

4.2 A



C_1 and C_2 are parallel:

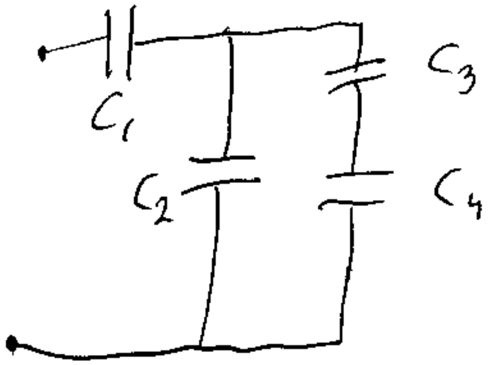
$$C_{12} = C_1 + C_2$$

The combination is in series with C_3:

$$\frac{1}{C} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{C_3 + C_{12}}{C_3 \cdot C_{12}}$$

$$\boxed{C = \frac{C_3 (C_1 + C_2)}{C_1 + C_2 + C_3}}$$

4.2 B



C_3 and C_4 in series:

$$C_{34} = \frac{C_3 \cdot C_4}{C_3 + C_4}$$

C_{34} in parallel to C_2:

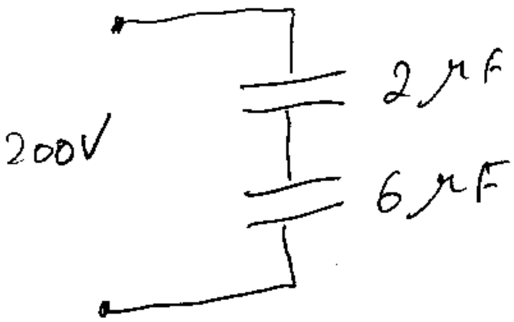
$$C_{234} = C_2 + C_{34} = C_2 + \frac{C_3 \cdot C_4}{C_3 + C_4} = \frac{C_2C_3 + C_2C_4 + C_3C_4}{C_3 + C_4}$$

C_{234} in series to C_1

$$C = \frac{C_1 \cdot C_{234}}{C_1 + C_{234}} = \frac{\dfrac{C_1C_2C_3 + C_1C_2C_4 + C_1C_3C_4}{C_3 + C_4}}{\dfrac{C_1C_3 + C_1C_4 + C_2C_3 + C_2C_4 + C_3C_4}{C_3 + C_4}}$$

$$\boxed{C = \frac{C_1C_2C_3 + C_1C_2C_4 + C_1C_3C_4}{C_1C_3 + C_1C_4 + C_2C_3 + C_2C_4 + C_3C_4}}$$

4.3 A



Capacitors in series have the same charge Q, voltages add.

$$C = \frac{2 \cdot 6}{2+6}\ \mu F = 1.5\ \mu F$$

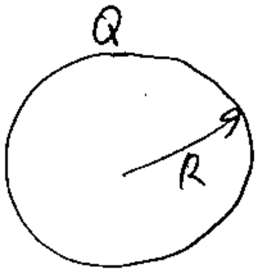
$$Q = C \Delta V = 1.5\ \mu F \cdot 200\ V = \underline{300\ \mu C}$$

$$V_1 = \frac{Q}{C_1} = \frac{300\ \mu C}{2\ \mu F} = \underline{150\ V}$$

$$V_2 = \frac{Q}{C_2} = \frac{300\ \mu F}{6\ \mu F} = \underline{50\ V}$$

$$V_1 + V_2 = 200\ V$$

4.6 A



There is no electric field inside, outside it is

$$E = \frac{Q}{4\pi \varepsilon_0 r^2}$$

$$u = \frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0 \frac{Q^2}{16\pi^2 \varepsilon_0^2 r^4}}{2} = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4}$$

The energy stored up to distance a is

$$U(a) = \int_R^a u \cdot 4\pi r^2 dr = \int_R^a \frac{Q^2}{8\pi \varepsilon_0 r^2} dr =$$

$$= \frac{Q^2}{8\pi \varepsilon_0} \left(\frac{1}{R} - \frac{1}{a}\right)$$

The total energy (up to ∞) is

$$\boxed{U_{tot} = \frac{Q^2}{8\pi \varepsilon_0 R}}$$

$\frac{U_{tot}}{2}$ is stored up to distance a;

$$\frac{Q^2}{16\pi \varepsilon_0 R} = \frac{Q^2}{8\pi \varepsilon_0} \left(\frac{1}{R} - \frac{1}{a}\right)$$

$$\frac{1}{2R} = \frac{1}{R} - \frac{1}{a}$$

$$\frac{1}{a} = \frac{1}{R} - \frac{1}{2R} = \frac{1}{2R}$$

$$\boxed{a = 2R}$$

4.6 B $R = 0.2\,m$, $V = 30000\,V$

a. From previous problem

$$U = \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$V = \frac{Q}{4\pi\varepsilon_0 R} \Rightarrow Q = 4\pi\varepsilon_0 R V$$

$$U = \frac{16\pi^2\varepsilon_0^2 R^2 V^2}{8\pi\varepsilon_0 R} = 2\pi\varepsilon_0 R V^2$$

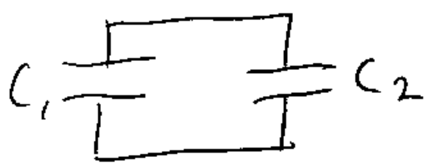
$$\boxed{U = 3.3 \cdot 10^{-7}\,J}$$

b. Each sphere will have $\frac{1}{2}$ of original charge, hence $\frac{1}{2}$ of original potential, hence $\frac{1}{4}$ of original energy ($U \propto V^2$).

But there are now two spheres, so the total energy will be $\frac{1}{2}$ of the original:

$$U = \frac{1}{2} \cdot 3.3 \cdot 10^{-7}\,J = \underline{1.7 \cdot 10^{-7}\,J}$$

4.6 C



Originally there is charge Q on C_1, hence energy:

$$U_0 = \frac{Q^2}{2C_1}$$

Afterwards they have charges Q_1 and Q_2, such that they have the same voltage:

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \implies Q_1 = \frac{C_1}{C_2} Q_2$$

But also

$$Q_1 + Q_2 = Q$$

$$\left(1+\frac{C_1}{C_2}\right)Q_2 = Q$$

$$\boxed{Q_2 = \frac{C_2}{C_1+C_2}Q} \qquad \boxed{Q_1 = \frac{C_1}{C_1+C_2}Q}$$

The energy:

$$U = U_1 + U_2 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} = \frac{C_1^{\cancel{2}}Q^2}{2\cancel{C_1}(C_1+C_2)^2} +$$

$$+ \frac{C_2^{\cancel{2}}Q^2}{2\cancel{C_2}(C_1+C_2)^2} = \frac{Q^2}{2(C_1+C_2)} < U_0$$

Difference

$$U_0 - U = \frac{Q^2}{2}\left(\frac{1}{C_1} - \frac{1}{C_1+C_2}\right) = \underline{\frac{Q^2C_2}{2C_1(C_1+C_2)}}$$

<u>4.60</u> $A = 500\ cm^2 = 0.05\ m^2$

$d = 1\ cm = 0.01\ m$

$V = 2000\ V$

a. $C = \frac{\varepsilon_0 A}{d} = 4.425\cdot 10^{-11} F$

6. $Q = CV = 8.85 \cdot 10^{-8}\ C$

After the sheet is inserted the capacitance becomes (see problem 4.10)

$$C' = C\frac{d}{d-a} = 1.25C$$

while the charge stays the same. The work done by electric forces is the difference between initial and final energies:

$$W = U - U' = \frac{Q^2}{2C} - \frac{Q^2}{2C'} = \frac{Q^2}{2}\left(\frac{1}{C} - \frac{1}{1.25C}\right) =$$

$$= 0.1\frac{Q^2}{C} = \underline{1.77 \cdot 10^{-6}\ J}$$

c, $V' = \frac{Q}{C'} = \frac{Q}{1.25C} = \frac{V}{1.25} = \underline{1600\ V}$

4.6 E If $d \to 2d$, the capacitance decreases by $\frac{1}{2}$: $C \to \frac{1}{2} C$. The charge stays the same, so

$$V \to 2V \qquad \left(V = \frac{Q}{C}\right)$$

$$\boxed{V_2 = 2V_1}$$

The energy stored doubles

$$\underline{U \to 2U} \qquad \left(U = \frac{Q^2}{2C}\right)$$

Another way to see this is the fact that the volume doubles while the electric field does not change. The plates attract each other, so there needs to be external work done to increase the separation. This is where the energy comes from.

4.7 A $E = \frac{V}{d}$ But each plate creates half of it:

$$E_1 = \frac{1}{2}\frac{V}{d}$$

The force on charge dq on one plate due to the field created by the other plate is

$$dF = E_1\, dq = \frac{1}{2}\frac{V}{d}\, dq$$

So the total force is

$$F = \frac{1}{2}\frac{V}{d}Q$$

where Q is the total charge on a plate

$$Q = CV = \frac{\varepsilon_0 A}{d}V$$

$$F = \frac{1}{2}\frac{V}{d}\frac{\varepsilon_0 A}{d}V = \underline{\frac{1}{2}\frac{\varepsilon_0 A V^2}{d^2}}$$