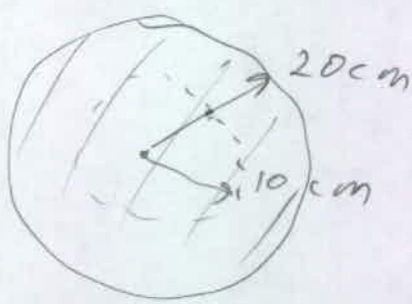
# Homework 2

2.1 D a. The electric field is 0 at the center of the sphere since the field created by oppositely located parts exactly cancel each other.

b. We use Gauss' theorem with a spherical surface of radius 10 cm = 0,1 m. The total charge inside is




$$Q_{in} = \rho V = \rho \frac{4}{3}\pi (0,1m)^3$$

$$\rho = \frac{Q}{\frac{4}{3}\pi(0,2m)^3}$$

$$Q_{in} = Q\left(\frac{0.1}{0.2}\right)^3 = \frac{Q}{8}$$

The surface area is

$$A = 4\pi (0{,}1m)^2$$

The flux is

$$\phi = AE = \frac{Q_{in}}{\varepsilon_0}$$

$$E = \frac{Q/8}{4\pi\varepsilon_0 (0{,}1m)^2} = \underline{1{,}125 \cdot 10^{11} Q \frac{N}{C}}$$

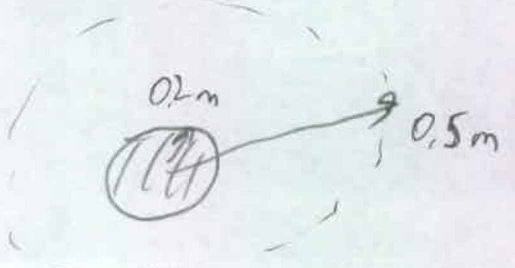
c. We use a similar method, but the Gaussian surface is the same as the surface of the sphere.

$$\phi = AE = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{\varepsilon_0 \cdot A} = \frac{Q}{4\pi\varepsilon_0 (0{,}2m)^2} = \underline{2{,}25 \cdot 10^{11} Q \frac{N}{C}}$$

d. The Gaussian surface has radius 50 cm = 0,5 m.

$$\phi = AE = \frac{Q}{\varepsilon_0}$$




$$E = \frac{Q}{\varepsilon_0 A} = \frac{Q}{4\pi\varepsilon_0 (0{,}5m)^2} = \underline{3{,}6 \cdot 10^{10} Q \frac{N}{C}}$$

<u>2.1 I</u> Let us first calculate the total charge inside a sphere of radius $r < R$:

$$Q_{in}(r) = \int_0^r 4\pi r^2 \rho(r)\, dr = 4\pi \int_0^r r^2 A(R-r)\, dr =$$

$$= 4\pi A \left( R\frac{r^3}{3} - \frac{r^4}{4} \right)\Big|_0^r = 4\pi A r^3 \left( \frac{R}{3} - \frac{r}{4} \right)$$

so the total charge is

$$Q = Q_{in}(R) = 4\pi A R^3 \left( \frac{R}{3} - \frac{R}{4} \right) = \frac{\pi A R^4}{3}$$

$$\boxed{A = \frac{3Q}{\pi R^4}}$$

We use Gauss' theorem to calculate the electric field at a distance $r$ from the center. The Gaussian surface is a sphere of radius $r$.

For <u>$r < R$</u>:

$$\phi = AE = 4\pi r^2 E = \frac{Q_{in}(r)}{\varepsilon_0}$$

$$E = \frac{Q_{in}(r)}{4\pi\varepsilon_0 r^2} = \underline{\frac{Ar}{\varepsilon_0}\left(\frac{R}{3} - \frac{r}{4}\right)}$$

For $\underline{r \geq R}$:

$$\phi = AE = 4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

$$\underline{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

3)


Let us consider first a spherical surface of radius $r$, $a < r \leq b$, and calculate the total charge inside. The part of volume that contains charge is

$$V = \frac{4}{3}\pi(r^3 - a^3)$$

so $$Q_{in}(r) = \frac{4}{3}\pi(r^3 - a^3)\rho$$

The total charge is

$$Q = Q_{in}(b) = \frac{4}{3}\pi(b^3 - a^3)\rho$$

$$\rho = \frac{3Q}{4\pi(b^3-a^3)}$$

We use Gauss' theorem as in the previous problems.

For $r \leq a$ the total charge inside a sphere of radius $r$ is 0 so the electric field is 0.

For $a < r \leq b$:

$$\Phi = AE = 4\pi r^2 E = \frac{Q_{in}(r)}{\varepsilon_0}$$

$$E = \frac{Q_{in}(r)}{4\pi\varepsilon_0 r^2} = \frac{\frac{4}{3}\cancel{\pi}(r^3-a^3)\rho}{\cancel{4\pi}\varepsilon_0 r^2} =$$

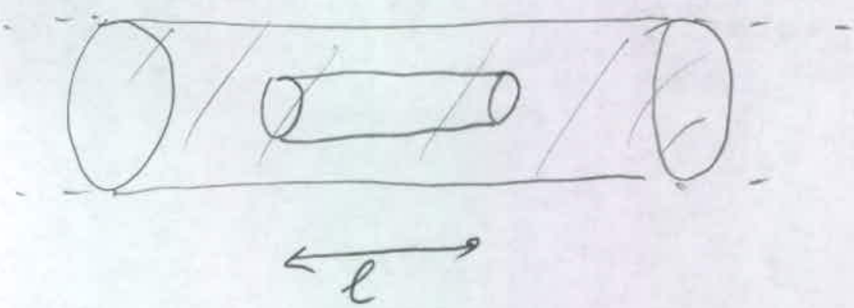
$$= \frac{(r^3-a^3)\cdot\frac{\cancel{3}Q}{4\pi(b^3-a^3)}}{\cancel{3}\varepsilon_0 r^2} = \frac{Q(r^3-a^3)}{4\pi\varepsilon_0 r^2(b^3-a^3)}$$

For $\underline{r > b}$:

$$\phi = AE = 4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

$$\underline{E = \frac{Q}{4\pi \varepsilon_0 r^2}}$$

4) $\rho = A(R-r)$



As a Gaussian surface we consider a cylinder of length $l$ and radius $r$ (for $r > R$ the cylinder will lie outside), There is no flux through the bases of the cylinder since the electric field is parallel to them.

For $\underline{r \leq R}$ the total charge inside is

$$Q_{in}(r) = \int_0^r 2\pi r \, \ell \, \rho(r) \, dr$$

To understand the expression above consider a cylindrical shell of radius r and thickness dr. The area of the base is $2\pi r \, dr$, so the volume is


$$2\pi r \, dr \cdot \ell$$

$$Q_{in}(r) = 2\pi \ell \int_0^r r A (R-r) \, dr = 2\pi \ell A \left( \frac{R r^2}{2} - \frac{r^3}{3} \right) \Bigg|_0^r =$$

$$= 2\pi \ell A r^2 \left( \frac{R}{2} - \frac{r}{3} \right)$$

Gauss' theorem;

$$\Phi = A E = 2\pi r \ell E = \frac{Q_{in}(r)}{\varepsilon_0}$$

$$E = \frac{Q_{in}(r)}{2\pi r \ell \varepsilon_0} = \frac{\cancel{2\pi} \ell A r^{\cancel{2}} \left( \frac{R}{2} - \frac{r}{3} \right)}{\cancel{2\pi} \cancel{r} \ell \varepsilon_0}$$

$$\boxed{E = \frac{A r}{\varepsilon_0} \left( \frac{R}{2} - \frac{r}{3} \right)}$$

For $r > R$ the total charge inside is the charge inside $R$:

$$Q_{in}(R) = 2\pi \ell A R^2 \left(\frac{R}{2} - \frac{R}{3}\right) = 2\pi \ell A \frac{R^3}{6}$$

$$\phi = AE = 2\pi r \ell E = \frac{Q_{in}(R)}{\varepsilon_0}$$

$$E = \frac{Q_{in}(R)}{2\pi r \ell \varepsilon_0} = \frac{\cancel{2\pi} \cancel{\ell} A \frac{R^3}{6}}{\cancel{2\pi} r \cancel{\ell} \varepsilon_0}$$

$$\boxed{E = \frac{A R^3}{6 r \varepsilon_0}}$$

5) If the charge density is uniform then for $\underline{r \leq R}$:

$$Q_{in}(r) = \rho V = \rho \pi r^2 \ell$$

$$\phi = AE = 2\pi r \ell E = \frac{Q_{in}(r)}{\varepsilon_0}$$

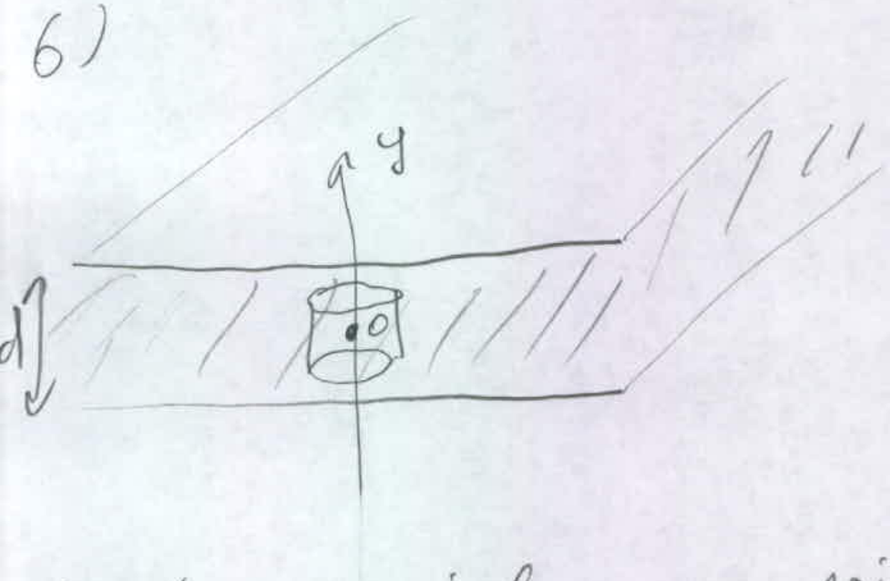
$$E = \frac{Q_{in}(r)}{2\pi r \ell \varepsilon_0} = \frac{\rho \cancel{\pi} r^{\cancel{2}} \cancel{\ell}}{2\cancel{\pi} \cancel{r} \cancel{\ell} \varepsilon_0} = \underline{\frac{\rho r}{2 \varepsilon_0}}$$

For $\underline{r > R}$:

$$Q_{in}(R) = \rho \pi R^2 \ell$$

$$E = \frac{Q_{in}(R)}{2\pi r \ell \varepsilon_0} = \frac{\rho \cancel{\pi} R^2 \cancel{\ell}}{2\cancel{\pi} r \cancel{\ell} \varepsilon_0} = \underline{\frac{\rho R^2}{2 r \varepsilon_0}}$$

6)




First consider a point with $\underline{|y| \leq \frac{d}{2}}$.
We use a cylinder with axis parallel to y and height $2|y|$ as our Gaussian surface. There is only flux through the bases of the cylinder since the electric field is parallel to y. The total charge inside is

$$Q_{in}(y) = \rho V = \rho A \cdot 2|y|$$

where A is the base area.

$$\phi = 2AE = \frac{Q_{in}(y)}{\varepsilon_0}$$

two bases

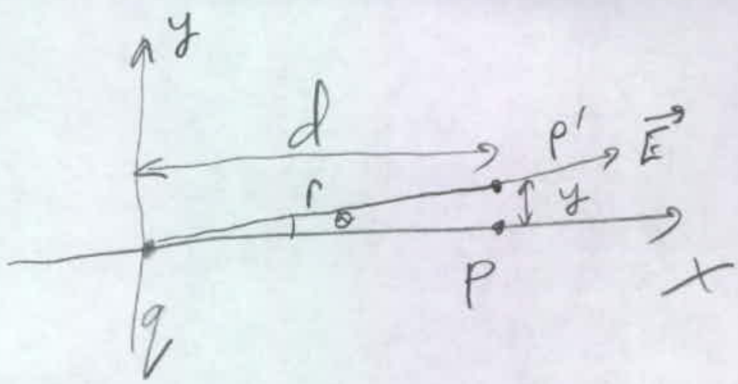
$$E = \frac{Q_{in}(y)}{2A\varepsilon_0} = \frac{\rho \cancel{A}\, \cancel{2}|y|}{\cancel{2}\cancel{A}\varepsilon_0} = \underline{\frac{\rho|y|}{\varepsilon_0}}$$

For $|y| > \frac{d}{2}$ there is charge only inside $\frac{d}{2}$, so

$$Q_{in}\left(\frac{d}{2}\right) = \rho A\, 2\frac{d}{2} = \rho A d$$

$$E = \frac{Q_{in}\left(\frac{d}{2}\right)}{2A\varepsilon_0} = \frac{\rho \cancel{A} d}{2\cancel{A}\varepsilon_0} = \underline{\frac{\rho d}{2\varepsilon_0}}$$

7)




At point P we have

$$E_x = \frac{1}{4\pi\varepsilon_0} \frac{q}{x^2} \quad (1) \quad \text{with} \quad x = d$$

Let us consider a point P' above P with $y << x$. At P'

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \simeq \frac{1}{4\pi\varepsilon_0} \frac{q}{x^2}$$

$$E_y = E \sin\theta \simeq E\,\theta \simeq E\frac{y}{x} \simeq \frac{1}{4\pi\varepsilon_0} \frac{qy}{x^3}$$

so $$\frac{\partial E_y}{\partial y}(P) = \frac{1}{4\pi\varepsilon_0} \frac{q}{x^3} = \frac{1}{4\pi\varepsilon_0} \frac{q}{d^3}$$

In exactly the same way

$$\frac{\partial E_z}{\partial z}(P) = \frac{1}{4\pi\varepsilon_0} \frac{q}{d^3}$$

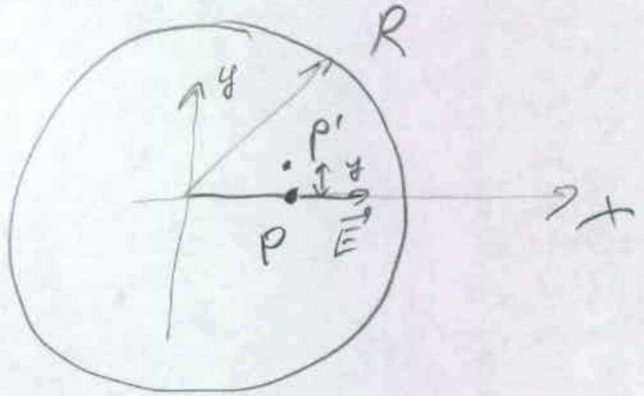
From (1):

$$\frac{\partial E_x}{\partial x}(P) = -2 \frac{1}{4\pi\varepsilon_0}\frac{q}{x^3} = -2\frac{1}{4\pi\varepsilon_0}\frac{q}{d^3}$$

Collecting everything:

$$div(E) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

8)




The electric field at a distance $r<R$ from the center is determined from Gauss' theorem

$$\phi = AE = 4\pi r^2 E = \frac{Q_{in}(r)}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0}$$

$$E = \frac{\rho r}{3\varepsilon_0}$$

This gives at point $P$

$$E_x = \frac{\rho x}{3\varepsilon_0} \qquad (1)$$

we construct a point $P'$ similar to the previous problem, At $P'$

$$E_y \simeq E\frac{y}{x} \simeq \frac{\rho x}{3\varepsilon_0}\frac{y}{x} = \frac{\rho y}{3\varepsilon_0} \qquad (2)$$

similarly

$$E_z \simeq \frac{\rho z}{3\varepsilon_0} \qquad (3)$$

Collecting (1), (2), and (3):

$$div(E) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} =$$

$$= \frac{\rho}{3\varepsilon_0} + \frac{\rho}{3\varepsilon_0} + \frac{\rho}{3\varepsilon_0}$$

$$div(E) = \frac{\rho}{\varepsilon_0}$$

Note that this is valid for any $r < R$, in particular $r = \frac{R}{2}$.