PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #5 SOLUTIONS
PRACTICE MIDTERM EXAM

(1) A nonrelativistic gas of spin-$\frac{1}{2}$ particles of mass m at temperature T and pressure p is in equilibrium with a surface. There is no magnetic field in the bulk, but the surface itself is magnetic, so the energy of an adsorbed particle is $-\Delta - \mu_0 H\sigma$, where $\sigma = \pm 1$ is the spin polarization and H is the surface magnetic field. The surface has $N_{\textsf{s}}$ adsorption sites.

(a) Compute the Landau free energy of the gas $\Omega_{\textsf{gas}}(T, V, \mu)$. Remember that each particle has two spin polarization states.

(b) Compute the Landau free energy of the surface $\Omega_{\textsf{surf}}(T, H, N_{\textsf{s}})$. Remember that each adsorption site can be in one of three possible states: empty, occupied with $\sigma = +1$, and occupied with $\sigma = -1$.

(c) Find an expression for the fraction $f(p, T, \Delta, H)$ of occupied adsorption sites.

(d) Find the surface magnetization, $M = \mu_0\big(N_{\textsf{surf},\uparrow} - N_{\textsf{surf},\downarrow}\big)$.

Solution :

(a) We have

$$\Xi_{\textsf{gas}}(T,V,\mu) = \sum_{N=0}^{\infty} e^{N\mu/k_{\rm B}T}\, Z(T,V,N) = \sum_{N=0}^{\infty} \frac{V^N}{N!}\, e^{N\mu/k_{\rm B}T}\, 2^N\, \lambda_T^{-3N}$$
$$= \exp\Big(2Vk_{\rm B}T\lambda_T^{-3}\, e^{\mu/k_{\rm B}T}\Big)\ ,$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_{\rm B}T}$ is the thermal wavelength. Thus,

$$\Omega_{\textsf{gas}} = -k_{\rm B}T \ln \Xi_{\textsf{gas}} = -2Vk_{\rm B}T\lambda_T^{-3}\, e^{\mu/k_{\rm B}T}\ .$$

(b) Each site on the surface is independent, with three possible energy states: $E = 0$ (vacant), $E = -\Delta - \mu_0 H$ (occupied with $\sigma = +1$), and $E = -\Delta + \mu_0 H$ (occupied with $\sigma = -1$). Thus,

$$\Xi_{\textsf{surf}}(T,H,N_{\textsf{s}}) = \Big(1 + e^{(\mu+\Delta+\mu_0 H)/k_{\rm B}T} + e^{(\mu+\Delta-\mu_0 H)/k_{\rm B}T}\Big)^{N_{\textsf{s}}}\ .$$

The surface free energy is

$$\Omega_{\textsf{surf}}(T,H,N_{\textsf{s}}) = -k_{\rm B}T\ln\Xi_{\textsf{surf}} = -N_{\textsf{s}}k_{\rm B}T\ln\Big(1 + 2\,e^{(\mu+\Delta)/k_{\rm B}T}\cosh(\mu_0 H/k_{\rm B}T)\Big)\ .$$

(c) The fraction of occupied surface sites is $f = \langle N_{\sf surf}/N_{\sf s}\rangle$. Thus,

$$f = -\frac{1}{N_{\sf s}}\,\frac{\partial \Omega_{\sf surf}}{\partial \mu} = \frac{2\,e^{(\mu+\Delta)/k_{\rm B}T}\cosh(\mu_0 H/k_{\rm B}T)}{1 + 2\,e^{(\mu+\Delta)/k_{\rm B}T}\cosh(\mu_0 H/k_{\rm B}T)} = \frac{2}{2 + e^{-(\mu+\Delta)/k_{\rm B}T}\,{\rm sech}(\mu_0 H/k_{\rm B}T)}\ .$$

To find $f(p,T,\Delta,H)$, we must eliminate μ in favor of p, the pressure in the gas. This is easy! From $\Omega_{\sf gas} = -pV$, we have $p = 2k_{\rm B}T\lambda_T^{-3}\,e^{\mu/k_{\rm B}T}$, hence

$$e^{-\mu/k_{\rm B}T} = \frac{2k_{\rm B}T}{p\,\lambda_T^3}\ .$$

Thus,

$$f(p,T,\Delta,H) = \frac{p\,\lambda_T^3}{p\,\lambda_T^3 + k_{\rm B}T\,e^{-\Delta/k_{\rm B}T}{\rm sech}(\mu_0 H/k_{\rm B}T)}\ .$$

Note that $f \to 1$ when $\Delta \to \infty$, when $T \to 0$, when $p \to \infty$, or when $H \to \infty$.

(d) The surface magnetization is

$$\begin{aligned} M = -\frac{\partial \Omega_{\sf surf}}{\partial H} &= N_{\sf s}\,\mu_0 \cdot \frac{2\,e^{(\mu+\Delta)/k_{\rm B}T}\sinh(\mu_0 H/k_{\rm B}T)}{1 + 2\,e^{(\mu+\Delta)/k_{\rm B}T}\cosh(\mu_0 H/k_{\rm B}T)} \\ &= \frac{N_{\sf s}\,\mu_0\,p\,\lambda_T^3\,\tanh(\mu_0 H/k_{\rm B}T)}{p\,\lambda_T^3 + k_{\rm B}T\,e^{-\Delta/k_{\rm B}T}{\rm sech}(\mu_0 H/k_{\rm B}T)}\ . \end{aligned}$$