PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #2

(**1**) Consider the matrix

$$M = \begin{pmatrix} 4 & 4 \\ -1 & 9 \end{pmatrix} .$$

(a) Find the characteristic polynomial $P(\lambda) = \mathsf{det}(\lambda\mathbb{I} - M)$ and the eigenvalues.

(b) For each eigenvalue λ_α, find the associated right eigenvector R^α_i and left eigenvector L^α_i. Normalize your eigenvectors so that $\langle\, L^\alpha \,|\, R^\beta \,\rangle = \delta_{\alpha\beta}$.

(c) Show explicitly that $M_{ij} = \sum_\alpha \lambda_\alpha \, R^\alpha_i \, L^\alpha_j$.

(**2**) A *Markov chain* is a probabilistic process which describes the transitions of discrete stochastic variables in time. Let $P_i(t)$ be the probability that the system is in state i at time t. The time evolution equation for the probabilities is

$$P_i(t+1) = \sum_j Y_{ij} \, P_j(t) \,.$$

Thus, we can think of $Y_{ij} = P(i \,,\, t+1 \,|\, j \,,\, t)$ as the *conditional probability* that the system is in state i at time $t{+}1$ given hat it was in state j at time t. Y is called the *transition matrix*. It must satisfy $\sum_i Y_{ij} = 1$ so that the total probability $\sum_i P_i(t)$ is conserved.

Suppose I have two bags of coins. Initially bag A contains two quarters and bag B contains five dimes. Now I do an experiment. Every minute I exchange a random coin chosen from each of the bags. Thus the number of coins in each bag does not fluctuate, but their values do fluctuate.

(a) Label all possible states of this system, consistent with the initial conditions. (*I.e.* there are always two quarters and five dimes shared among the two bags.)

(b) Construct the transition matrix Y_{ij}.

(c) Show that the total probability is conserved is $\sum_i Y_{ij} = 1$, and verify this is the case for your transition matrix Y. This establishes that $(1, 1, \ldots, 1)$ is a left eigenvector of Y corresponding to eigenvalue $\lambda = 1$.

(d) Find the eigenvalues of Y.

(e) Show that as $t \to \infty$, the probability $P_i(t)$ converges to an equilibrium distribution $P^{\rm eq}_i$ which is given by the right eigenvector of i corresponding to eigenvalue $\lambda = 1$. Find $P^{\rm eq}_i$, and find the long time averages for the value of the coins in each of the bags.

$(\mathbf{3})$ Poincar'e recurrence is guaranteed for phase space dynamics that are *invertible*, *volume preserving*, and acting on a *bounded phase space*.

(a) Give an example of a map which is volume preserving on a bounded phase space, but which is not invertible and not recurrent.

(b) Give an example of a map which is invertible on a bounded phase space, but which is not volume preserving and not recurrent.

(c) Give an example of a map which is invertible and volume preserving, but on an unbounded phase space and not recurrent.

$(\mathbf{4})$ Consider a toroidal phase space $(x, p) \in \mathbb{T}^2$. You can describe the torus as a square $[0,1] \times [0,1]$ with opposite sides identified. Design your own modified Arnold cat map acting on this phase space, *i.e.* a 2×2 matrix with integer coefficients and determinant 1.

(a) Start with an initial distribution localized around the center – say a disc centered at $(\frac{1}{2}, \frac{1}{2})$. Show how these initial conditions evolve under your map. Can you tell whether your dynamics are mixing?

(b) Now take a pixelated image. For reasons discussed in the lecture notes, this image should exhibit Poincaré recurrence. Can you see this happening?

$(\mathbf{5})$ Consider a spin singlet formed by two $S = \frac{1}{2}$ particles, $|\,\Psi\,\rangle = \frac{1}{\sqrt{2}}\big(|\!\uparrow_{\mathsf{A}}\downarrow_{\mathsf{B}}\rangle - |\!\downarrow_{\mathsf{A}}\uparrow_{\mathsf{B}}\rangle\big)$. Find the reduced density matrix, $\rho_{\mathsf{A}} = \mathsf{Tr}_{\mathsf{B}}\,|\,\Psi\,\rangle\langle\,\Psi\,|$.