PHYSICS 140A : STATISTICAL PHYSICS
HW ASSIGNMENT #1

(**1**) Consider the contraption in Fig. 1. At each of k steps, a particle can fork to either the left ($n_j = 1$) or to the right ($n_j = 0$). The final location is then a k-digit binary number.

(a) Assume the probability for moving to the left is p and the probability for moving to the right is $q \equiv 1 - p$ at each fork, independent of what happens at any of the other forks. *I.e.* all the forks are uncorrelated. Compute $\langle X_k \rangle$. *Hint: X_k can be represented as a k digit binary number, i.e.* $X_k = n_{k-1} n_{k-2} \cdots n_1 n_0 = \sum_{j=0}^{k-1} 2^j n_j$.

(b) Compute $\langle X_k^2 \rangle$ and the variance $\langle X_k^2 \rangle - \langle X_k \rangle^2$.

(c) X_k may be written as the sum of k random numbers. Does X_k satisfy the central limit theorem as $k \to \infty$? Why or why not?

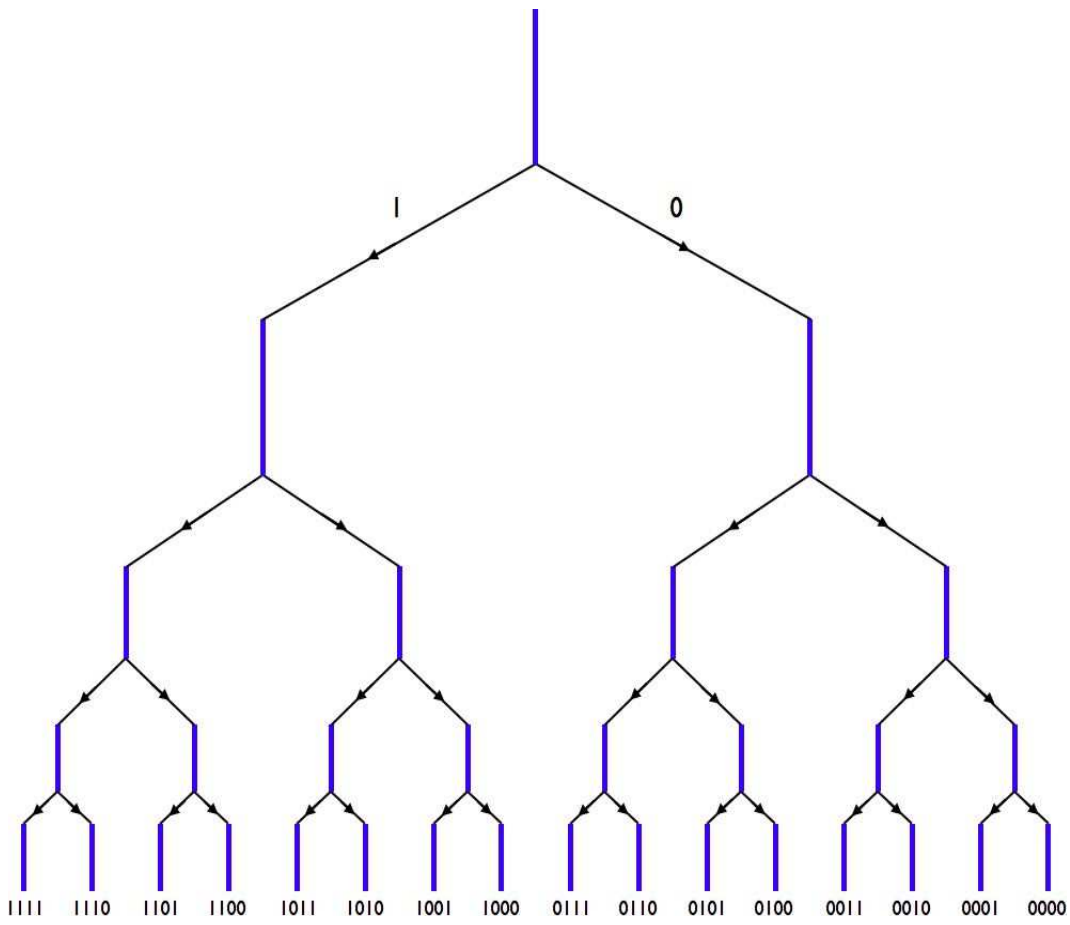

Figure 1: Generator for a k-digit random binary number ($k = 4$ shown).

($\mathbf{2}$) Let $P(x) = (2\pi\sigma^2)^{-1/2}\, e^{-(x-\mu)^2/2\sigma^2}$. Compute the following integrals:

(a) $I = \int\limits_{-\infty}^{\infty} dx\; P(x)\, x^3$.

(b) $I = \int\limits_{-\infty}^{\infty} dx\; P(x)\; \cos(Qx)$.

(c) $I = \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy\; P(x)\, P(y)\, e^{\kappa^2 xy}$. You may set $\mu = 0$ to make this somewhat simpler. Under what conditions does this expression converge? (Here κ has units of $1/x$.)

($\mathbf{3}$) The binomial distribution,

$$B_N(n,p) = \binom{N}{n}\, p^n\, (1-p)^{N-n} \; ,$$

tells us the probability for n successes in N trials if the individual trial success probability is p. The average number of successes is $\nu = \sum_{n=0}^{N} n\, B_N(n,p) = Np$. Consider the limit $N \to \infty$.

(a) Show that the probability of n successes becomes a function of n and ν alone. That is, evaluate

$$P_\nu(n) = \lim_{N\to\infty} B_N(n, \nu/N) \; .$$

This is the *Poisson distribution*.

(b) Show that the moments of the Poisson distribution are given by

$$\langle n^k \rangle = e^{-\nu} \left(\nu \, \frac{\partial}{\partial \nu} \right)^k e^{\nu} \; .$$

(c) Evaluate the mean and variance of the Poisson distribution.

The Poisson distribution is also known as the *law of rare events* since $p = \nu/N \to 0$ in the $N \to \infty$ limit. See `http://en.wikipedia.org/wiki/Poisson_distribution#Occurrence` for some amusing applications of the Poisson distribution.

($\mathbf{4}$) Consider a D-dimensional *random walk* on a hypercubic lattice. The position of a particle after N steps is given by

$$\boldsymbol{R}_N = \sum_{j=1}^{N} \hat{\boldsymbol{n}}_j \; ,$$

where $\hat{\boldsymbol{n}}_j$ can take on one of $2D$ possible values: $\hat{\boldsymbol{n}}_j \in \big\{ \pm\, \hat{\boldsymbol{e}}_1, \ldots, \pm\hat{\boldsymbol{e}}_D \big\}$, where $\hat{\boldsymbol{e}}_\mu$ is the unit vector along the positive x_μ axis. Each of these possible values occurs with probability $1/2D$, and each step is statistically independent from all other steps.

(a) Consider the *generating function* $S_N(\boldsymbol{k}) = \langle e^{i\boldsymbol{k}\cdot\boldsymbol{R}_N} \rangle$. Show that

$$\langle R_N^{\alpha_1} \cdots R_N^{\alpha_J} \rangle = \frac{1}{i}\frac{\partial}{\partial k_{\alpha_1}} \cdots \frac{1}{i}\frac{\partial}{\partial k_{\alpha_J}}\bigg|_{\boldsymbol{k}=0} S_N(\boldsymbol{k}) \ .$$

For example, $\langle R_N^\alpha R_N^\beta \rangle = -\big(\partial^2 S_N(\boldsymbol{k})/\partial k_\alpha \partial k_\beta\big)_{\boldsymbol{k}=0}$.

(b) Evaluate $S_N(\boldsymbol{k})$ for the case $D = 3$ and compute the quantities $\langle X_N^4 \rangle$ and $\langle X_N^2\, Y_N^2 \rangle$.

(**5**) A rare disease is known to occur in $f = 0.02\%$ of the general population. Doctors have designed a test for the disease with $\nu = 99.9\%$ sensitivity and $\rho = 99.95\%$ specificity.

(a) What is the probability that someone who tests positive for the disease is actually sick?

(b) Suppose the test is administered twice, and the results of the two tests are independent. If a random individual tests positive both times, what are the chances he or she actually has the disease?

(c) For a binary partition of events, find an expression for $P(X|A \cap B)$ in terms of $P(A|X)$, $P(B|X)$, $P(A|\neg X)$, $P(B|\neg X)$, and the priors $P(X)$ and $P(\neg X) = 1 - P(X)$. You should assume A and B are independent, so $P(A \cap B|X) = P(A|X) \cdot P(B|X)$.

(**6**) Compute the entropy in the F08 Physics 140A grade distribution (in bits). See

http://physics.ucsd.edu/students/courses/fall2008/physics140/

for the distribution itself. You should assume 11 possible grades: A+, A, A-, B+, B, B-, C+, C, C-, D, F.