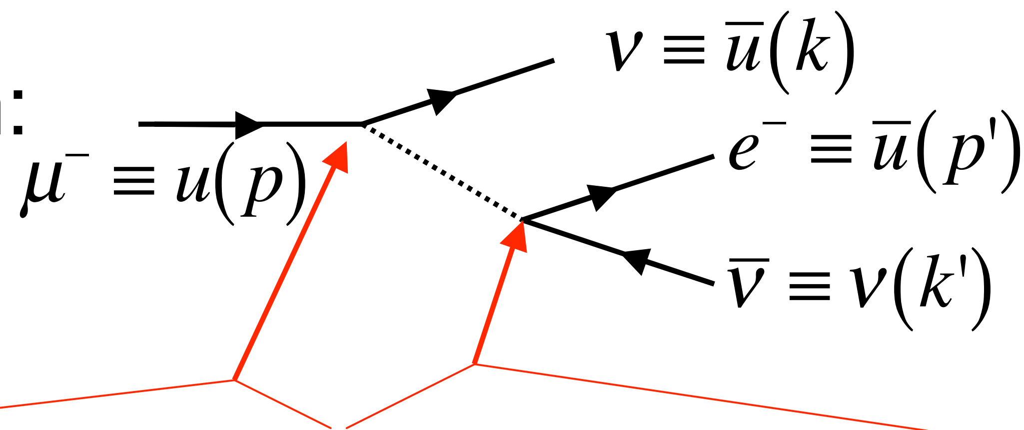
Physics 222 UCSD/225b UCSB

Lecture 3

- Weak Interactions (continued)
  - muon decay
  - Pion decay

Muon Decay Overview (1)

- Feynman diagram:



- Matrix Element:

$$M = \frac{G}{\sqrt{2}}\left[\overline{u}(k)\gamma^{\mu}\left(1-\gamma^{5}\right)u(p)\right]\left[\overline{u}(p')\gamma_{\mu}\left(1-\gamma^{5}\right)v(k')\right]$$

G = effective coupling of a 4-fermion interaction

Structure of 4-fermion interaction is (V-A)x(V-A)

We clearly want to test this experimentally,
e.g. compare against (V-A)x(V+A), S, P, T, etc.

Muon Decay Overview (2)

- Differential width:

$$d\Gamma = \frac{1}{2E}\overline{|M|^2}\,dQ$$

- Phase space differential:

$$dQ = \frac{d^3 p'}{(2\pi)^3 2E'}\frac{d^3 k}{(2\pi)^3 2\omega}\frac{d^3 k'}{(2\pi)^3 2\omega'}(2\pi)^4\,\delta^{(4)}(p - p' - k - k')$$

Calculational Challenges

- There's a spin averaged matrix element involved, requiring the use of some trace theorems.
- The phase space integral is not trivial.

I'll provide you with an outline of how these are done, and leave it up to you to go through the details in H&M.

Phase Space Integration (1)

- We have:

$$dQ = \frac{d^3p'}{(2\pi)^3 2E'} \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'} (2\pi)^4 \delta^{(4)}(p - p' - k - k')$$

- We can get rid of one of the three d^3p/E by using:

$$\int \frac{d^3k}{\omega} = \int d^4k \;\; \theta(\omega)\delta(k^2)$$

- And eliminating the d^4k integral against the 4d delta-function. This leads to:

$$dQ = \frac{1}{(2\pi)^5} \frac{d^3p'}{2E'} \frac{d^3k'}{2\omega'} (2\pi)^4 \theta(E - E' - \omega')\delta\left((p - p' - k')^2\right)$$

This means $E - E' - \omega' > 0$

Phase Space Integration (2)

- As was done for beta-decay, we replace:

$$d^3p'\,d^3k' = 4\pi E'^2\,dE'\,2\pi\omega'^2\,d\omega'\,d\cos\theta$$

- And evaluate delta-fct argument in muon restframe:

$$\delta\left((p-p'-k')^2\right) = \delta\left(m^2 - 2mE' - 2m\omega' + 2E'\omega'(1-\cos\theta)\right)$$

Recall: primed variables are from second W vertex.

Spin Average Matrix Element

- We neglect the mass of the electron and neutrinos.
- And use the trace theorem H&M (12.29) to arrive at:

$$\overline{|M|^2} = \frac{1}{2}\sum_{spins}|M|^2 = 64G^2(k \cdot p')(k' \cdot p)$$

Muon Restframe

- To actually do the phase space integral, we go into the muon restframe where we find:
  - $2\,kp' = (k+p')^2 = (p-k')^2 = m^2 - 2pk' = m^2 - 2m\,\omega'$

    Mass of e and nu are ignored

    4-mom. cons.

    Mass nu is ignored

    Mu restframe
  - $k'p = \omega' m$
- And as a result we get:

$$\overline{|M|^2} = 32G^2\left(m^2 - 2m\omega'\right)m\omega'$$

Putting the pieces together and doing the integration over cosθ, the opening angle of e and its anti-neutrino, we arrive at:

$$d\Gamma = \frac{G^2}{2\pi^3} dE'\, d\omega'\, m\omega(m - 2\omega')$$

$$\frac{1}{2}m - E' \leq \omega' \leq \frac{1}{2}m$$

$$0 \leq E' \leq \frac{1}{2}m$$

The inequality come from the requirement that cosθ is physical. And are easily understood from the allowed 3-body phasespace where one of the 3 is at rest.

Electron Energy Spectrum

- Integrate over electron antineutrino energy:

$$\frac{d\Gamma}{dE'} = \frac{G^2}{2\pi^3} \int_{\frac{1}{2}m-E'}^{\frac{1}{2}m} d\omega' \, m\omega (m - 2\omega')$$

$$\frac{d\Gamma}{dE'} = \frac{G^2}{2\pi^3} m^2 E'^2 \left(3 - \frac{4E'}{m}\right)$$

$$0 \leq E' \leq \frac{1}{2}m$$

The spectrum can be used to test V-A. This is discussed as "measurement of Michel Parameters" in the literature.

Michel Parameters

- It can be shown that any 4-fermion coupling will lead to an electron spectrum like the one we derived here, once we allow a "*Michel Parameter*" ρ, as follows: $x = \frac{2E_e}{m_\mu}$

$$\frac{1}{\Gamma}\frac{d\Gamma}{dx} = 12x^2\left[1 - x + \frac{2}{3}\rho\left(\frac{4}{3}x - 1\right)\right]$$

Measured Value:

$$\rho_\mu = 0.7509 \pm 0.0010$$
$$\rho_\tau = 0.745 \pm 0.008$$

- ρ=0 for (V-A)x(V+A),S,P; ρ=1 for T
- ρ=0.75 for (V-A)x(V-A)
- With polarized muon beams and measurement of electron polarization, other "*Michel Parameters*" come into play.

Total Decay Width of Muon

- Integrate over electron energy:

$$\Gamma = \frac{1}{\tau} = \frac{G^2}{2\pi^3} m^2 \int_0^{\frac{1}{2}m} dE' \, E'^2 \left(3 - \frac{4E'}{m} \right)$$

$$\Gamma = \frac{G^2 m^5}{192\pi^3}$$

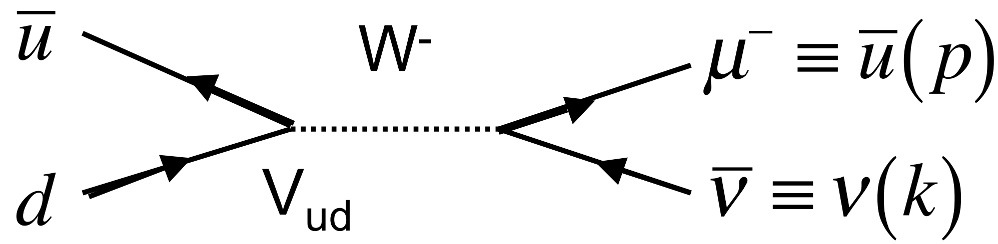
$$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$$

$$\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau$$

$$\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau$$

Note: Comparing muon and tau decays, as well as tau decays to electron and muon, allows for stringent tests of lepton universality to better than 1%.

Pion Decay



- Leptonic vertex is identical to the leptonic current vertex in muon decay.
- Hadronic vertex needs to be parametrized as it can NOT be treated as a current composed of free quarks.

Parametrization of Hadronic Current

- Matrix element is Lorentz invariant scalar.
  - Hadronic current must be vector or axial vector
- Pion is spinless
  - Q is the only vector to construct a current from.
- The current at the hadronic vertex thus must be of the form:

$$q^\mu f_\pi(q^2) = q^\mu f_\pi(m_\pi^2) = q^\mu f_\pi$$

- However, as $q^2 = m_\pi^2$ = constant, we refer to f_π simply as the "*pion decay constant*".
- All other purely leptonic decays of weakly decaying mesons can be calculated in the same way. There are thus "*decay constants*" for B^0, B_s^0, D^+, D_s^+, K^+,etc.

Aside:

- This sort of parametrization is “reused” also when extrapolating from semileptonic to hadronic decays at fixed q^2
  - E.g. Using B -> D lnu to predict B -> D X where X is some hadron.
  - This is crude, but works reasonably well in some cases.

Matrix Element for Pion Decay

$$M = \frac{GV_{ud}}{\sqrt{2}}\left(p^{\mu} + k^{\mu}\right) f_{\pi}\left[\overline{u}(p)\gamma_{\mu}\left(1-\gamma^{5}\right)v(k)\right]$$

Now use the Dirac Equation for muon and neutrino:

$$\overline{u}(p)\left(p^{\mu}\gamma_{\mu} - m_{\mu}\right) = 0$$

=> $\boxed{\overline{u}(p)p^{\mu}\gamma_{\mu}}\left(1-\gamma^{5}\right)v(k) = \boxed{\overline{u}(p)m_{\mu}}\left(1-\gamma^{5}\right)v(k)$

$$k^{\mu}\gamma_{\mu}v(k) = 0$$

=> $\overline{u}(p)k^{\mu}\gamma_{\mu}\left(1-\gamma^{5}\right)v(k) = 0$

Note: this works same way for any aV+bA .

=> $M = \frac{G}{\sqrt{2}} m_{\mu} f_{\pi}\left[\overline{u}(p)\left(1-\gamma^{5}\right)v(k)\right]$

Trace and Spin averaging

- The spin average matrix element squared is then given by:

$$\overline{|M|^2} = |V_{ud}|^2 \frac{G^2}{2} f_\pi^2 m_\mu^2 Tr\left[\left(p^\mu \gamma_\mu + m_\mu\right)\left(1-\gamma^5\right) k^\mu \gamma_\mu \left(1+\gamma^5\right)\right]$$

$$\overline{|M|^2} = 4G^2 |V_{ud}|^2 f_\pi^2 m_\mu^2 (p \cdot k)$$

- You can convince yourself that this trace is correct by going back to H&M (6.19), (6.20). The only difference is the "+" sign. This comes from "pulling" a gamma matrix past gamma5.

Going into the pion restframe

- We get: $$p \cdot k = E\omega - \vec{p}\vec{k} = \omega\left(E + \omega\right)$$

- Where we used that muon and neutrino are back to back in the pion restframe.

Pion leptonic decay width

- Putting it all together, we then get:

$$d\Gamma = \frac{1}{2m_\pi}\overline{|M|^2}\frac{d^3p}{(2\pi)^3 2E}\frac{d^3k}{(2\pi)^3 2\omega}(2\pi)^4\delta(q-p-k)$$

$$\Gamma = \frac{G^2|V_{ud}|^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi}\int\frac{d^3p}{E}\frac{d^3k}{\omega}\delta(m_\pi - E - \omega)\delta^{(3)}(\vec{k}+\vec{p})\omega(E+\omega)$$

Energy conservation

3-momentum conservation
Use this to kill int over d^3p

Pion leptonic width

- I'll spare you the details of the integrations. They are discussed in H&M p.265f
- The final result is:

$$\Gamma = \frac{G^2 |V_{ud}|^2}{8\pi} f_\pi^2 \boxed{m_\mu^2} m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

Helicity Suppression

Helicity Suppression

- The pion has spin=0 .
- Angular momentum is conserved.
  - ⇒ Electron and anti-neutrino have same helicity.
  - ⇒ However, weak current does not couple to J=0 electron & antineutrino pair.
  - ⇒ Rate is suppressed by a factor: $\frac{m_\mu^2}{m_\pi^2}$

$$\Gamma_\pi = \frac{G^2 |V_{ud}|^2}{8\pi} f_\pi^2 m_\pi^3 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \times \frac{m_\mu^2}{m_\pi^2}$$

$$\Gamma_\mu = \frac{G^2}{192\pi^3} m_\mu^5$$

Helicity suppression

Experimentally

- As the pion decay constant is not known, it is much more powerful to form the ratio of partial widths:

$$\frac{\Gamma\left(\pi^- \to e^- \overline{\nu}_e\right)}{\Gamma\left(\pi^- \to \mu^- \overline{\nu}_\mu\right)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.233 \times 10^{-4}$$

Experimentally, we find: (1.230 +- 0.004) x 10^{-4}

Aside: Theory number here includes radiative corrections !!!
I.e., this is not just the mass ratio as indicated !!!

Experimental Relevance

- We've encountered this a few times already, and now we have actually shown the size of the helicity suppression, and where it comes from.

$$\frac{\Gamma\left(\pi^- \to e^- \overline{\nu}_e\right)}{\Gamma\left(\pi^- \to \mu^- \overline{\nu}_\mu\right)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.23 \times 10^{-4}$$

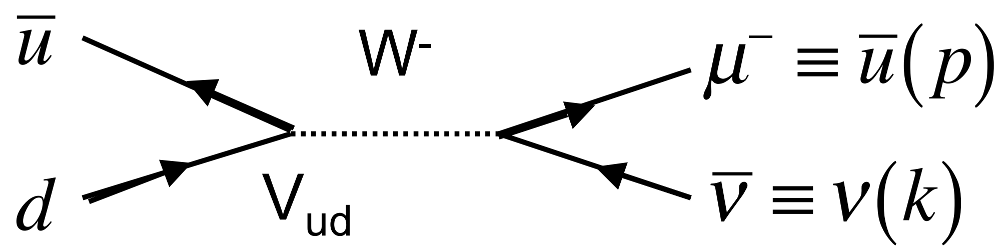
Accordingly, pion decay produces a rather pure muon neutrino beam, with the charge of the pion determining neutrino or anti-neutrino in the beam.

Origin of Helicity Suppression Recap

- The muon mass entered because of the vector nature of the leptonic current.
  - ⇒Either V or A or some combination of aV+bA will all lead to helicity suppression.
  - ⇒In particular, a charged weak current with S,P, or T instead of V,A is NOT consistent with experiment.
- In addition, we used:
  - – Neutrinos are massless
  - – Electron-muon universality

Window for New Physics via leptonic decays

Example
B^+ decay



$$\Gamma = \frac{G^2 |V_{ub}|^2}{8\pi} f_B^2 m_\mu^2 m_B \left(1 - \frac{m_\mu^2}{m_B^2}\right)^2$$

Helicity Suppression

The smallness of V_{ub} and muon mass allows for propagators other than W to compete, especially if they do not suffer from helicity suppression => e.g. charged Higgs