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Magnetic Buoyancy and the Boussinesq Approximation

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The full Boussinesq equations for hydromagnetic convection are derived and shown to include the effects of magnetic buoyancy. Instabilities caused by magnetic buoyancy are analyzed and their roles in double convection are brought out.

1. INTRODUCTION

Strong magnetic fields are generated in the convective zones of the sun and other late-type stars. This process cannot be properly described without an understanding of the interaction between magnetic fields, convection and rotation (Moffatt, 1978; Parker, 1979; Acheson, 1979a, b). So far, however, the effects of thermal and magnetic buoyancy have been treated separately. Magnetic inhibition of thermal convection in a Boussinesq fluid has been investigated in some detail (e.g. Chandrasekhar, 1961; Proctor and Weiss, 1982) but magnetic buoyancy has been less thoroughly explored. Parker (1966, 1979) and Gilman (1970) considered the linear stability of an isothermal gas, and Acheson (1978, 1979a) has provided a local stability analysis for a thermally stratified layer. Most nonlinear studies have been concerned with the behaviour of isolated flux tubes (Parker, 1979).

To simplify the incorporation of magnetic buoyancy into the study of hydromagnetic convection we shall obtain the appropriate equations in the Boussinesq approximation. This is readily done by following the same physical assertions as are used to extract the usual Boussinesq equations from those for a fully compressible gas (Jeffreys, 1930; Spiegel and Veronis, 1960). An alternative approach is to use formal expansions (Malkus, 1964). In this way, for a special choice of scaling in linear theory, Roberts and Stewartson (1977) have derived a similar set of equations in a mathematically systematic, but physically less illuminating, fashion.

Once the Boussinesq equations are available, the competition between thermal and magnetic buoyancy can take its proper place as an example of double convection (Spiegel, 1972). Indeed, we show here that in the simplest case, when the field lines remain uncurved, the equations become identical to those that govern thermohaline convection (Turner, 1973; Huppert, 1977). We also describe the destabilization of hydromagnetic waves by magnetic buoyancy (Gilman, 1970; Acheson, 1978, 1979a) within the Boussinesq approximation.

In the next section we obtain the Boussinesq equations for hydromagnetic convection in a gas. The basic physical idea is that the layer under study is so thin that variations of thermodynamic quantities across it are very slight, and that all relevant time scales are long compared to the time for acoustic waves to cross the region. For a horizontal layer of depth d, we assume that d is very small compared to the temperature scale height, H_T, the pressure scale height, H_p and the density scale height H_ρ.

These requirements are not always sufficient to ensure that the mean properties of the fluid do not vary greatly from place to place. If the horizontal gradients of material properties are small and comparable to the vertical gradients, there is a risk of nonBoussinesq behaviour whenever large horizontal scales of motion arise. An example of such a breakdown of the Boussinesq conditions occurs in convection with flux fixed on the boundaries because the critical horizontal wave number for instability is small (Depassier and Spiegel, 1981, 1982). For that case, an additional requirement is needed to render the Boussinesq approximation qualitatively valid, namely that the horizontal scale of motion be sufficiently small. In the case of a polytropic gas, the Boussinesq approximation holds for motions of small but finite amplitude when the horizontal scale is much less than the geometric mean of d and H_p. We assume here that this requirement is met. Also, we consider only the strong form of the Boussinesq approximation, in which the gas constant, R_*, the specific heat at constant pressure, C_p, the permeability, μ, the magnetic diffusivity, η, the thermal conductivity, K, and the viscosity $\rho\nu$ are all taken as constant.

A further condition is needed. Let a subscript zero indicate a typical value of a quantity in the fluid, so that ρ_0 (for instance) is a constant to be judiciously chosen, equal to the value of the density in the fluid at some point in space and time. We then require that the fluid velocity $\mathbf{u}$ should

remain highly subsonic, so that $\mathbf{u}_0^2 \ll v_s^2 \equiv \gamma p_0/\rho_0$, and that the magnetic field $\mathbf{B}$ is sufficiently weak for the Alfvén speed to be much less than the speed of sound, so that $v_A^2 \equiv B_0^2/\mu\rho_0 \ll v_s^2$. Then the fast magnetoacoustic wave, which is purely longitudinal and travels isotropically at the sound speed v_s, remains distinct from the slow wave and the pure Alfvén wave, both of which are purely transverse and travel at the Alfvén speed v_A along uniform fields.

2. THE BOUSSINESQ APPROXIMATION

The basic equations for a gas in a uniform gravitational field, in the magnetohydrodynamic approximation, are

$$\rho[\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u}] = -g\rho\hat{\mathbf{z}} - \nabla p + \rho\nu[\nabla^2\mathbf{u} + \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{u})] + \mu^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}, \quad (1)$$

$$\partial\rho/\partial t = -\nabla\cdot(\rho\mathbf{u}), \quad (2)$$

$$\partial\mathbf{B}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} - \mathbf{B}\nabla\cdot\mathbf{u} + \eta\nabla^2\mathbf{B}, \quad (3)$$

$$\nabla\cdot\mathbf{B} = 0, \quad (4)$$

$$\rho C_p[\partial T/\partial t + (\mathbf{u}\cdot\nabla)T] - [\partial p/\partial t + (\mathbf{u}\cdot\nabla)p]$$
$$= K\nabla^2 T + \text{dissipative heating terms}, \quad (5)$$

and

$$p = R_*\rho T, \quad (6)$$

where $\hat{\mathbf{z}}$ is a unit vector in the upward vertical direction.

Let us suppose that the imposed magnetic field is horizontal; then equations (1), (3), (4) and (5) admit a static solution $\rho = \bar{\rho}(z)$, $p = \bar{p}(z)$, $T = \bar{T}(z)$, $\mathbf{B} = \bar{\mathbf{B}}(z)$ with $\bar{T}$ and $\bar{\mathbf{B}}$ linear in the vertical coordinate, z. Now let $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, $T = \bar{T} + \delta T$, $\mathbf{B} = \bar{\mathbf{B}} + \delta\mathbf{B}$, where $\delta\rho$ etc. are assumed small. Then the continuity equation (2) can be written in the form

$$\nabla\cdot\mathbf{u} = -(\mathbf{u}\cdot\nabla)(\ln\bar{\rho}) + O(\delta\rho/\rho_0). \quad (7)$$

Substituting from (7) into the induction equation (3), we have

$$\partial\mathbf{B}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} - w\mathbf{B}/H_\rho + \eta\nabla^2\mathbf{B}, \quad (8)$$

approximately, where $\mathbf{u}=(u,v,w)$ and the density scale height $H_\rho = -(d\ln\bar{\rho}/dz)^{-1}$.

Since the first term on the right of (7) is $O(d/H_\rho)$ compared to $\nabla\cdot\mathbf{u}$, the equation of continuity reduces to

$$\nabla\cdot\mathbf{u}=0. \tag{9}$$

In the equation of motion the density is regarded as constant, and equal to ρ_0, everywhere except in the buoyancy term. Hence we replace (1) by

$$\rho_0[\partial\mathbf{u}/\partial t+(\mathbf{u}\cdot\nabla)\mathbf{u}]=-g\rho\hat{\mathbf{z}}-\nabla\Pi+\rho_0\nu\nabla^2\mathbf{u}+\mu^{-1}(\mathbf{B}\cdot\nabla)\mathbf{B}, \tag{10}$$

where the total pressure

$$\Pi=p+p_m\equiv p+\mathbf{B}^2/2\mu. \tag{11}$$

We set $\Pi=\bar{\Pi}+\delta\Pi$ where, in the absence of motion, the gradient of total pressure is $d\bar{\Pi}/dz=-g\rho_0$. Next we adopt a scaling such that $|\nabla\delta\Pi|\sim g\delta\rho$. Then

$$\delta\Pi/p_0\sim(d/H_p)(\delta\rho/\rho_0), \tag{12}$$

where the pressure scale height $H_p=(p_0/g\rho_0)$. Thus the variation in *total* pressure remains small. Now the linearized equation of state becomes

$$(\delta\rho/\rho_0)=-(\delta T/T_0)+(\delta p/p_0). \tag{13}$$

In the normal Boussinesq approximation, with no magnetic field, we neglect δp in the equation of state but the basic idea of magnetic buoyancy is that δp can no longer be ignored, since it is $\delta\Pi$ that is negligible. Hence $\delta p\approx-\delta p_m=-(B^2-\bar{B}^2)/(2\mu)$ and

$$(\delta\rho/\rho_0)=-(\delta T/T_0)-\delta p_m/p_0. \tag{14}$$

We shall be concerned with situations where the thermal and magnetic contributions to the density are comparable in magnitude.

This argument can be extended to other equations of state having the form $\rho=\rho(T,p)$. On expanding about the reference state, we have

$$\rho=\rho_0+(\partial_T\rho)_0(T-\bar{T})+(\partial_p\rho)_0(p-\bar{p})+\ldots. \tag{15}$$

Since $\delta\Pi$ remains small, it follows that

$$\delta\rho/\rho_0\simeq-\alpha_T\delta T-\kappa_T\delta p_m, \tag{16}$$

where α_T is the coefficient of thermal expansion and κ_T is the isothermal compressibility. In a liquid, where κ_T is small, large gradients in $\bar{\mathbf{B}}$ are needed to make magnetic buoyancy significant.

Finally, we consider the energy equation (5). The ratio of the viscous and ohmic dissipation terms to the thermal diffusion term on the right-hand side of (5) is of order d/H_T, where the temperature scale height $H_T=C_pT_0/g$. On the left side of (5) we can write

$$Dp/Dt\equiv\partial p/\partial t+\mathbf{u}\cdot\nabla p\approx-wg\rho_0-D(\delta p_m)/Dt. \tag{17}$$

Thus (5) becomes

$$D[\delta T+(C_p\rho_0)^{-1}\delta p_m]/Dt+w\beta=\kappa\nabla^2\delta T, \tag{18}$$

where the thermal diffusivity $\kappa=K/(C_p\rho_0)$ and the *subadiabatic* temperature gradient

$$\beta=T_0\gamma^{-1}d[\ln(p\rho^{-\gamma})]/dz, \tag{19}$$

with $\gamma=C_p/C_v$. Finally, we form the scalar product of equation (8) with $\mathbf{B}$ to obtain

$$D(\delta p_m)/Dt=-\mathbf{B}_0^2w\mu^{-1}d[\ln(\bar{B}\bar{\rho}^{-1})]/dz+\mu^{-1}\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{u}+\eta\nabla^2\mathbf{B}]. \tag{20}$$

Equations (4), (8), (9), (10), (14), (18) and (20) govern hydromagnetic convection in a gas in the magnetoBoussinesq approximation. As is apparent from (14), they describe both thermal and magnetic buoyancy. A further simplification (cf. Fricke, 1969) leads to the equations commonly used to describe convection in the presence of an initially uniform magnetic field (Chandrasekhar, 1961): when $|\delta p_m|\ll R_*\rho_0|\delta T|$, magnetic buoyancy is generally unimportant and the terms involving δp_m in (14) and (18) may be ignored. There is then no need to retain the penultimate term in (8).

3. THE INTERCHANGE INSTABILITY

The significance of magnetic buoyancy in the Boussinesq equations is best grasped by studying simple examples. Consider an unstably stratified horizontal field such that

$$\bar{\mathbf{B}} = B_0(1-\zeta z/d)\hat{\mathbf{y}}, \tag{21}$$

where $\zeta \ll 1$, so that the magnetic scale height $H_B = d/\zeta \gg d$, and $\hat{\mathbf{y}}$ is a unit vector in the y-direction. We expect that the fluctuating field will be small so that $\delta p_m \simeq B_0 b/\mu$, where $b = \delta\mathbf{B}\cdot\hat{\mathbf{y}}$. In this section, we consider the simplest possible configuration, with purely two-dimensional motion restricted to directions parallel to the xz-plane and purely horizontal magnetic fields with $(\mathbf{B}\cdot\nabla)\mathbf{B} = 0$.

3.1 Adiabatic interchanges

In the absence of any dissipation, we have, from (18) and (20),

$$D[\delta T + (C_p\rho_0)^{-1}\delta p_m]/Dt + w\beta = 0, \quad D(\delta p_m)/Dt + w\alpha = 0, \tag{22}$$

where

$$\alpha = B_0^2\mu^{-1} d[\ln(\bar{B}/\bar{\rho}^{-1})]/dz. \tag{23}$$

We introduce a stream function ψ, with

$$\mathbf{u} = (-\partial\psi/\partial z, 0, \ \partial\psi/\partial x). \tag{24}$$

Then the y-component of the curl of (10) gives

$$D(\nabla^2\psi)/Dt - g\partial_x(T_0^{-1}\delta T + p_0^{-1}\delta p_m) = 0. \tag{25}$$

For perturbations with $\psi \propto \sin(lx/d)\sin(\pi z/d)\exp(st)$ it follows that

$$s^2 = -gl^2(l^2+\pi^2)^{-1}[(\gamma p_0)^{-1}\alpha + T_0^{-1}\beta]. \tag{26}$$

Thus the configuration is unstable if

$$d[\ln(\bar{p}/\bar{\rho}^\gamma) + (B_0^2/\mu p_0)\ln(\bar{B}/\bar{\rho})]/dz < 0. \tag{27}$$

This generalized Schwarzschild criterion is a special case of that derived by Schubert (1968, equation (32)) and subsequently discussed by Cadez (1974), Acheson (1979a) and Schatten and Sofia (1981). It can also be obtained by considering simple interchanges (Tayler, 1973; Moffatt, 1978; Acheson and Gibbons, 1978; Acheson, 1979a).

3.2. Effects of diffusion

The equations become more complicated in the presence of diffusion. From (20), we have

$$D(\delta p_m)/Dt + w\alpha = \eta\nabla^2\delta p_m, \tag{28}$$

while, from (18) and (28), we find that

$$D(\delta T^*)/Dt + w\beta^* = \kappa\nabla^2\delta T^*, \tag{29}$$

where

$$\delta T^* = \delta T - \tau\delta p_m/[C_p\rho_0(1-\tau)], \quad \beta^* = \beta - \alpha/[C_p\rho_0(1-\tau)], \tag{30}$$

and $\tau = \eta/\kappa$. In order to write the equations in dimensionless form, we adopt d as the unit of length and, for this section only, we take d^2/κ as the unit of time. It is convenient to define a modified dimensionless temperature Θ and a dimensionless magnetic field Σ such that

$$\Theta = \delta T^*/(|\beta^*|d), \quad \Sigma = \delta p_m/(|\alpha|d). \tag{31}$$

We then obtain the dimensionless equations

$$(\partial_t - \sigma\nabla^2)\nabla^2\psi = \sigma R_T\partial_x\Theta + \sigma R_B\partial_x\Sigma + \partial(\psi,\nabla^2\psi)/\partial(x,z), \tag{32}$$

$$(\partial_t - \nabla^2)\Theta + \hat{\beta}\partial_x\psi = \partial(\psi,\Theta)/\partial(x,z), \tag{33}$$

$$(\partial_t - \tau\nabla^2)\Sigma + \hat{\alpha}\partial_x\psi = \partial(\psi,\Sigma)/\partial(x,z), \tag{34}$$

where

$$\hat{\alpha} = \operatorname{sgn}\alpha, \quad \hat{\beta} = \operatorname{sgn}\beta^*, \tag{35}$$

with $\sigma = \nu/\kappa$ and

$$R_T = \frac{gd^4}{\kappa\nu T_0}|\beta^*|, \quad R_B = \frac{\gamma-\tau}{\gamma(1-\tau)}\frac{gd^4}{\kappa\nu p_0}|\alpha|. \tag{36}$$

Equations (32)–(34) are now identical in form to those describing two-dimensional thermosolutal convection. Hydromagnetic convection with magnetic buoyancy included is thus a paradigm of two-dimensional double convection. For convenience, we consider here the idealized boundary conditions that have been extensively employed in the literature (e.g. Chandrasekhar, 1961; Roberts and Stewartson, 1977). These "free" boundary conditions permit the introduction of sines and cosines as eigenfunctions of the linear problem. The astrophysically relevant case is that with stable thermal but unstable magnetic stratification ($\beta=-1$, $\epsilon=1$, $\tau<1$), which corresponds to the production of salt fingers in experiments on thermohaline convection (Turner, 1973). The static configuration is unstable to modes of the form $\psi=\psi_0 \sin(lx)\sin(\pi z)$ if

$$\tau^{-1}R_B > (l^2+\pi^2)^3 l^{-2} + R_T \geqq (27/4)\pi^4 + R_T. \tag{37}$$

In astrophysically interesting situations, $\kappa \gg \eta \gg \nu$ ($\sigma \ll \tau \ll 1$) and instability arises when

$$-B_0^2(\mu p_0)^{-1}\, d[\ln(\bar{B}/\bar{\rho})]/dz > \eta\kappa^{-1}\, d[\ln(\bar{p}/\bar{\rho}^{\gamma})]/dz \tag{38}$$

(Schubert, 1968). This agrees well with the criterion obtained by Acheson (1978, 1979a; note that his thermal diffusivity is our $\gamma\kappa$). Thus the interchange instability, in which flux tubes are exchanged without bending, is described by the Boussinesq approximation. The nonlinear development of this instability should lead to the formation of narrow magnetic flux sheets, analogous to salt fingers. However, it turns out that three-dimensional instabilities, in which magnetic buoyancy is complicated by curvature of the magnetic field, may be more important in practice.

4. THE UNDULAR INSTABILITY

Hydromagnetic waves become unstable owing to magnetic buoyancy. The problem is more complicated than the one just discussed, and it illustrates nicely the advantages of the Boussinesq approximation in magnetoconvection. We shall confine ourselves to a linear treatment of the stability of a perfect fluid that is adiabatically stratified. The results closely resemble those first obtained for an isothermal gas by Gilman (1970) and subsequently generalized by Acheson (1978, 1979a).

4.1 Nonrotating system

We set all diffusivities equal to zero, and adopt the Alfvén period $d/v_A = (\mu\rho_0)^{1/2} d/B_0$ as unit of time. The static configuration is identical to the one considered in the previous section. Since the gas is homentropic, $\beta=0$ and

so, from (18), $\delta T=\delta p_m/(C_p\rho_0)$. To first order in perturbation amplitudes, the dimensionless equations therefore become

$$\partial_t \delta\mathbf{B} = (1-\zeta z)\partial_y \mathbf{u} + (\zeta-\xi)w\hat{\mathbf{y}}, \tag{39}$$

$$\partial_t \mathbf{u} = -\nabla\delta\Pi + \xi(\delta\mathbf{B}\cdot\hat{\mathbf{y}})\hat{\mathbf{z}} + \partial_y\delta\mathbf{B} - \zeta(\delta\mathbf{B}\cdot\hat{\mathbf{z}})\hat{\mathbf{y}}, \tag{40}$$

where $\zeta=d/H_p=d/\gamma H_p$. As before, we assume idealized boundary conditions at $z=0, 1$, but now we consider waves propagating along the magnetic field, so that $w \propto \cos(lx)\sin(\pi z)\exp[i(my-\omega t)]$, for example. It turns out that the most unstable modes have a very long wavelength and so we shall assume that $m^2 \ll l^2, \pi^2$ (cf. Roberts and Stewartson, 1977). For sufficiently small m, that is $O(\xi)$, we can then introduce stream and flux functions, ψ and χ, such that

$$\mathbf{u}=\left(-\frac{\partial\psi}{\partial z}, v, \frac{\partial\psi}{\partial x}\right), \quad \delta\mathbf{B}=\left(-\frac{\partial\chi}{\partial z}, b, \frac{\partial\chi}{\partial x}\right), \tag{41}$$

where

$$\psi/\psi_0=\chi/\chi_0=\sin(lx)\sin(\pi z)\exp[i(my-\omega t)], \tag{42}$$

$$b/b_0=v/v_0=\cos(lx)\sin(\pi z)\exp[i(my-\omega t)]. \tag{43}$$

From Eq. (39) we find that

$$\omega\chi_0+m\psi_0=0, \quad \omega b_0+mv_0-il(\zeta-\xi)\psi_0=0. \tag{44}$$

The pressure perturbation $\delta\Pi$ can be calculated by forming the divergence of (40). It can easily be verified that $\partial_y\delta\Pi=O[m^2/(l^2+\pi^2)]$ and is therefore negligible. Hence the y-component of the equation of motion gives

$$\omega v_0+mb_0+i\zeta l\chi_0=0, \tag{45}$$

while, from the y-component of the curl of (40), we derive

$$\omega\psi_0+m\chi_0-[il\xi/(l^2+\pi^2)]b_0=0. \tag{46}$$

From Eqs. (44)–(46) we obtain the dispersion relation

$$\omega^4-\omega^2\left[2m^2-\frac{l^2}{l^2+\pi^2}(\zeta-\xi)\xi\right]+m^2\left[m^2-\frac{l^2}{l^2+\pi^2}\zeta\xi\right]=0. \tag{47}$$

Instability therefore occurs through an exponentially growing mode for

$$m^2 < [l^2/(l^2+\pi^2)]\zeta\xi < \zeta\xi. \tag{48}$$

Hence the configuration is unstable if

$$-(g\rho_0/\gamma p_0)d(\ln \bar{B})/dz > (l^2+\pi^2)m^2/d^2l^2, \tag{49}$$

(Acheson, 1979a). For sufficiently small m, instability occurs if $|\bar{\mathbf{B}}|$ decreases with height.

Gilman (1970) considered perturbations to an isothermal equilibrium. He assumed $\kappa \gg \eta, \nu$ and restricted his attention to modes with $l^2 \gg \pi^2$, so that temperature fluctuations could be ignored, but he did not make the Boussinesq approximation. The resulting dispersion relation reduces to (47) in the limit when $v_A^2 \ll v_s^2$, supporting the validity of our equations. Note that the most unstable modes are elongated parallel to the field and narrow in the x-direction, with a corkscrew motion such that $v^2 \gg w^2 \gg u^2$. Instability persists as $m \to 0$; only for $m=0$ does (47) give the criterion for instability to interchanges.

4.2 Rapidly rotating system

Now consider a system rotating with a fixed angular velocity $\mathbf{\Omega} = (-\Omega, 0, 0)$ about the x-axis (like an equatorial portion of a star). In the sun, rotation is sufficiently rapid to stabilize interchange instabilities locally, and undular instabilities are much altered (Acheson and Gibbons, 1978; Acheson, 1979a). The wave that becomes unstable is the slow magnetostrophic mode, with a time scale, $\Omega d^2/(m^2 v_A^2)$, much longer than the rotation period (e.g. Moffatt, 1978). Following Moffatt, in the magnetostrophic approximation, we replace Eq. (40) by

$$2\mathbf{\Omega} \times \mathbf{u} = -\nabla\delta\Pi + \zeta b\hat{\mathbf{z}} + \partial_y \delta\mathbf{B} - \zeta(\delta\mathbf{B}\cdot\hat{\mathbf{z}})\hat{\mathbf{y}}. \tag{50}$$

Use of (42) and (43) in the y-components of (50) and of its curl gives

$$2\Omega l\psi_0 = \mathrm{im}\, b_0 - \zeta l\chi_0, \quad 2\Omega l v_0 = -\xi l b_0 - \mathrm{im}\,(l^2+\pi^2)\chi_0, \tag{51}$$

and, from (44) and (51), we obtain the dispersion relation

$$4\Omega^2\omega^2 - 4\Omega m\xi\omega - m^2[l^{-2}(l^2+\pi^2)m^2 - \zeta\xi] = 0. \tag{52}$$

Instability occurs in the form of exponentially growing waves when ω becomes complex, that is for

$$m^2 < l^2\xi(\zeta-\xi)/(l^2+\pi^2). \tag{53}$$

Thus the rotating layer is unstable if

$$-(g\rho_0/\gamma p_0)d[\ln(\bar{B}/\bar{\rho})]/dz > (l^2+\pi^2)m^2/d^2l^2, \tag{54}$$

(Acheson 1979a); for sufficiently small m there is instability if $|\bar{B}/\bar{\rho}|$ decreases with height. These results are again consistent with those derived for an isothermal gas ($\gamma=1$) if $v_A^2 \ll v_s^2$ (Moffatt, 1978; Acheson and Gibbons, 1978).

5. MAGNETOCONVECTION AND MAGNETIC BUOYANCY

We have shown that magnetic buoyancy can be accommodated within the framework of an extended Boussinesq approximation. In the astrophysically relevant limit, when $v_A^2 \ll v_s^2$, the results obtained by Gilman (1970) and Moffatt (1978) can all be recovered. Acheson's (1978, 1979a) local analysis goes beyond that in Section 4 to include the effects of diffusion and stratification; D. W. Hughes (private communication) has rederived Acheson's dispersion relation for a plane layer using the Boussinesq approximation. The local approximation and the magnetoBoussinesq approximation with periodic boundary conditions are equivalent in linear theory. The advantages of the Boussinesq approximation are that it facilitates the inclusion of other boundary conditions (Childress and Spiegel, 1981) and yields simplified equations that are valid in the nonlinear regime. We find that it clarifies the physics of magnetic buoyancy. We also anticipate that the introduction of a solenoidal velocity will ease the numerical treatment of nonlinear magnetoconvection and so afford some hope of understanding the effects of diffusion on the undular instability of finite amplitudes.

We need not repeat here the summary of the magnetoBoussinesq equations given at the end of Section 2, but we must add one observation on the essential difference between that system and the ordinary Boussinesq description of convection. In the latter, we simply replace the statement $\nabla\cdot\mathbf{u} = O(d/H_\rho)$ by the condition that $\mathbf{u}$ should be solenoidal. Since the pressure does not appear in the equation of state, it is at our disposal in the momentum equation to help ensure that $\nabla\cdot\mathbf{u}=0$; indeed we need some condition to determine the pressure. In Boussinesq magnetoconvection, the condition $\nabla\cdot\mathbf{B}=0$ enters in a different way. We can satisfy $\nabla\cdot\mathbf{u}=0$ exactly, even though it is only an approximation, but we can satisfy $\nabla\cdot\mathbf{B}=0$ only approximately, though it is supposed to be exact. In both cases, we make a slight error and we have to be sure that the size of the error does not invalidate the equations affected to the order of interest. We know that this works for $\nabla\cdot\mathbf{u}$ but we need to verify that all is well for $\nabla\cdot\mathbf{B}=0$.

The problem of breaking the solenoidal constraint on $\mathbf{B}$ arises only when magnetic buoyancy is important. The divergence of (8) can be written

$$(\partial_t - w/H_p)\nabla\cdot\mathbf{B} = H_\rho^{-1}\mathbf{B}\cdot\nabla w. \tag{55}$$

In the cases considered in Sections 3 and 4, the right-hand side of this equation is zero or $O(d^2/H_\rho^2)$. Then the error that is introduced into the induction equation by the failure of the magnetic field to be exactly solenoidal is smaller than the error expected in the magnetoBoussinesq approximation itself. However, it is clear that there is no guarantee that this remains true for times much longer than H_p/v_A, and $\nabla\cdot\mathbf{B}$ should be monitored in time dependent problems. When introducing nonBoussinesq corrections one should take care to ensure that a similar self-consistency obtains in each order. If magnetic buoyancy is unimportant, we need not be concerned with such details, as the ordinary Boussinesq approximation suffices for most purposes.

The right side of (55) would be significant if the scale of variation of w in the direction of $\mathbf{B}$ were comparable with the layer depth. It turns out, however, that magnetic buoyancy is not important in such circumstances; to demonstrate this we have first to clarify the way in which magnetic pressure affects the Boussinesq equation of state (13). Let us make a Helmholtz decomposition of the Lorentz force into irrotational and solenoidal parts:

$$\mu^{-1}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla P + \nabla\times\mathbf{Q}, \quad \nabla\cdot\mathbf{Q}=0. \tag{56}$$

We may impose the boundary conditions

$$\partial P/\partial n = -\mu^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}\cdot\mathbf{n}, \quad \mathbf{Q}=0, \tag{57}$$

where $\mathbf{n}$ is a unit vector normal to the boundary. Then P and $\mathbf{Q}$ are obtained by solving

$$\nabla^2 P = -\mu^{-1}\nabla\cdot[(\nabla\times\mathbf{B})\times\mathbf{B}], \quad \nabla^2\mathbf{Q} = -\mu^{-1}\nabla\times[(\nabla\times\mathbf{B})\times\mathbf{B}], \tag{58}$$

subject to (57). Forming the divergence and the curl of the Boussinesq equation of motion (10) yields

$$\nabla^2(\delta p + \delta P) = -g\partial_z\delta\rho - \rho_0\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}], \tag{59}$$

$$\nabla^2(\delta\mathbf{Q}) = -g\nabla\delta\rho\times\hat{\mathbf{z}} + \rho_0[\nu\nabla^2\boldsymbol{\omega} - \partial_t\boldsymbol{\omega} + \nabla\times(\mathbf{u}\times\boldsymbol{\omega})], \tag{60}$$

where the vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{u}$. At the upper and lower boundaries, where $w=0$, the vertical component of the equation of motion reduces to

$$\partial_z(\delta p + \delta P) = -g\delta\rho. \tag{61}$$

Now the right-hand sides of (59) and (61) are both of order d/H_p so, to leading order in d/H_p,

$$\nabla^2(\delta p + \delta P) = 0, \tag{62}$$

and

$$\partial_z(\delta p + \delta P) = 0 \quad \text{on} \quad z=0, d. \tag{63}$$

Hence $\delta p + \delta P =$ constant and we may take

$$\delta p = -\delta P. \tag{64}$$

In Section 2, we adopted a different decomposition of the Lorentz force:

$$\mu^{-1}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla p_m + (\mathbf{B}\cdot\nabla)\mathbf{B}. \tag{65}$$

From (58) and (57) it follows that the magnetic pressure $p_m = P$ only if

$$\nabla\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}] = 0. \tag{66}$$

and

$$(\mathbf{B}\cdot\nabla)\mathbf{B}\cdot\hat{\mathbf{z}} = 0 \quad \text{on} \quad z=0, d. \tag{67}$$

For certain simple boundary conditions, for example if $B_z=0$ or if $B_x=B_y=0$, condition (67) is met. For the two-dimensional interchanges discussed in Section 3, $(\mathbf{B}\cdot\nabla)\mathbf{B}=0$ and (66) is trivially satisfied so that $P=p_m$. For the undular instabilities of Section 4

$$|(\mathbf{B}\cdot\nabla)\mathbf{B}|/|\nabla\delta p_m| = O(d/H_p). \tag{68}$$

Hence $\delta P = \delta p_m + O(d/H_p)$. Now magnetic buoyancy becomes important when $\delta p = -\delta P = O(p_0 \delta\rho_0/\rho_0)$, but from (60) we would expect that

$$|\delta \mathbf{Q}|/p_0 = O[(d/H_p)(\delta\rho/\rho_0)]. \quad (69)$$

Equation (60) can be rewritten as

$$\rho_0 D\boldsymbol{\omega}/Dt = \rho_0(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \rho_0 \nu \nabla^2 \boldsymbol{\omega} - g\nabla\delta\rho \times \hat{\mathbf{z}} - \nabla^2(\delta \mathbf{Q}). \quad (70)$$

Within the Boussinesq approximation we can recognize three different regimes, depending on the importance of the magnetic torque, $-\nabla^2(\delta \mathbf{Q})$ in (70). (1) In the first regime, the magnetic torque cannot be neglected. If there is a balance between the magnetic torque and the buoyant torque then

$$|\delta P| \sim |\delta \mathbf{Q}| = O[(d/H_p)p_0\delta\rho/\rho_0]. \quad (71)$$

It follows from (59) that $|\delta p/p_0| \ll |\delta\rho/\rho_0|$. This is the regime of magnetoconvection where motion is driven by thermal buoyancy and magnetic buoyancy is unimportant. (2) If, on the other hand, we seek a regime in which magnetic buoyancy is important then it follows from (59) that $|\delta \mathbf{Q}|/|\delta P| = O(d/H_p)$ and the magnetic torque in (70) can be neglected. In practice, this means that (68) is satisfied and so $\delta p \simeq -\delta p_m$. Magnetic buoyancy is significant if the wavelength of disturbances in the direction parallel to the imposed field is comparable to the scale height. Thus the right-hand side of (55) is indeed $O(d^2/H_\rho^2)$. (3) Finally, there is a regime with rapidly varying motion (involving hydromagnetic waves) in which the magnetic torque is balanced by $\rho_0 D\boldsymbol{\omega}/Dt$ in (70). Then $|\delta P| \sim |\delta \mathbf{Q}| = O(p_0\delta\rho/\rho_0)$ and the waves may be either damped or destabilized by the effects of thermal and magnetic stratification.

The Boussinesq approximation can be used to explore the influence of magnetic fields on motion driven by buoyancy in stars. In this approximation, instabilities driven by thermal and magnetic buoyancy provide problems that are complementary to each other; in particular solutions describing convection in a vertical magnetic field will hardly be affected by the addition of magnetic buoyancy. In a stellar convection zone, with d/H_p of the order of unity, this distinction disappears. Furthermore, at the surface of a star like the sun, where magnetic flux is confined to isolated tubes with $p_m \sim p_0$ (Parker, 1979), convection cannot be adequately represented in the Boussinesq approximation.

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Convective Dynamos with Intermediate and Strong Fields

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The weak-field Benard-type dynamo treated by Soward is considered here at higher levels of the induced magnetic field. Two sources of instability are found to occur in the intermediate field regime $M \sim T^{1/12}$, where M and T are the Hartmann and Taylor numbers. On the time scale of magnetic diffusion, solutions may blow up in finite time owing to destabilization of the convection by the magnetic field. On a faster time scale a dynamic instability related to MAC-wave instability can also occur. It is therefore concluded that the asymptotic structure of this dynamo is unstable to virtual increases in the magnetic field energy.

In an attempt to model stabilization of the dynamo in a strong-field regime we consider two approximations. In the first, a truncated expansion in three-dimensional plane waves is studied numerically. A second approach utilizes an *ad hoc* set of ordinary differential equations which contains many of the features of convection dynamos at all field energies. Both of these models exhibit temporal intermittency of the dynamo effect.

1. INTRODUCTION

Recent attempts to provide a dynamical basis for the geodynamo have focused on systems which involve both convection and rapid rotation. Childress and Soward (1972) suggested that classical Benard convection in a rotating plane layer would, in the conducting case, lead to dynamo action, and noted that the relevant asymptotic structure of a stable